

**Computer Science Department  
Stanford University  
Comprehensive Examination in Numerical Analysis  
Autumn 1994**

**October 21, 1994**

***READ THIS FIRST!***

- 1. You should write your answers for this part of the Comprehensive Examination in a BLUE BOOK. Be sure to write your MAGIC NUMBER on the cover of every blue book that you use.**
- 2. This exam is CLOSED BOOK. You may not use notes, articles, or books.**
- 3. Show your work, since PARTIAL CREDIT will be given for incomplete answers. For example, you can get credit for making a reasonable start on a problem even if the idea doesn't work out; you can also get credit for realizing that certain approaches are incorrect. On a true/false question, you might get partial credit for explaining why you think something is true when it is actually false. But no partial credit can be given if you write nothing.**

**Comprehensive Examination: Numerical Analysis (30 points). Fall 1994**

**(Problem I)**

(15 points). **Linear Systems and Matrices**

(a) Define the condition number  $\mathcal{K}$  of a matrix  $A$ . If

$$Ax = b, \quad Ay = b + r$$

state the upper and lower bounds relating the relative error  $\frac{\|x-y\|}{\|x\|}$  to the relative perturbation  $\frac{\|r\|}{\|b\|}$ . Briefly discuss the implications of these inequalities for the numerical solution of linear systems.

(b) Let  $A$  be an  $m \times m$  symmetric matrix with eigenvalues  $\lambda_i$  and corresponding eigenvectors  $\phi_i$ . What are the important properties of the  $\lambda_i$  and  $\phi_i$  which follow from  $A$  being symmetric.

(c) Let  $\|\cdot\|$  denote the Euclidean vector norm and also use the same notation for the induced matrix norm. By noting that any vector can be expressed as a linear combination of the  $\phi_i$ , prove that

$$\|A\| = \max_{1 \leq i \leq m} |\lambda_i|.$$

(d) Calculate  $\mathcal{K}(A)$  in the Euclidean norm for a symmetric positive definite matrix.

**(Problem II)**

(10 points). **Quadrature**

(a) Define the composite trapezoidal rule for the approximate integration of

$$I := \int_a^b f(x) dx$$

over  $n$  intervals of equal length  $h = (b-a)/n$ . State the magnitude of the error in terms of  $h$ , under an assumption on the smoothness of  $f$  which you should state.

(b) Show the weights  $\{w_i\}_{i=1}^n$  and nodes  $\{x_i\}_{i=1}^n$  in the Gaussian quadrature formula

$$\int_{-1}^1 f(x) dx \approx \sum_{j=1}^n w_j f(x_j)$$

satisfy

$$\sum_{j=1}^n w_j (x_j)^i = 0, \quad i = 1, 3, \dots, 2n-1$$

and

$$\sum_{j=1}^n w_j (x_j)^i = \frac{2}{i+1}, \quad i = 0, 2, \dots, 2n-2.$$