

**Computer Science Department  
Stanford University  
Comprehensive Examination in Numerical Analysis  
Autumn 1998**

**October 30, 1998**

***READ THIS FIRST!***

1. You should write your answers for this part of the Comprehensive Examination in a **BLUE BOOK**. Be sure to write your **MAGIC NUMBER** on the cover of every blue book that you use.
  
2. This exam is **CLOSED BOOK**. You may not use notes, articles, or books.
  
3. Show your work, since **PARTIAL CREDIT** will be given for incomplete answers. For example, you can get credit for making a reasonable start on a problem even if the idea doesn't work out; you can also get credit for realizing that certain approaches are incorrect. On a true/false question, you might get partial credit for explaining why you think something is true when it is actually false. But no partial credit can be given if you write nothing.

**Comprehensive Exam: Numerical Analysis (30 points). Fall 1998**

**(Solution I)**

**(16 points). Rootfinding for Nonlinear Equations**

(a) Define the *order of convergence* of a sequence  $\{x_n | n \geq 0\}$  to a point  $\alpha$ . When is the convergence said to be *linear*? If the convergence is linear, define the *rate of linear convergence*.

(b) Given a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and two points  $a, b : f(a)f(b) < 0$  define the *bisection method*, in algorithmic form, to find a root  $\alpha$  in  $[a, b]$  satisfying  $f(\alpha) = 0$ .

Let  $c_1 = [a + b]/2$  and let  $\{c_n | n \geq 1\}$  be the sequence of approximations to  $\alpha$  generated by the method. Show that, for some root  $\alpha$ ,

$$|\alpha - c_n| \leq \frac{|b - a|}{2^n}.$$

What can be said about the rate of linear convergence in this case?

(c) Given a twice continuously differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  define Newton iteration to generate a sequence  $\{x_n | n \geq 0\}$  to locate a root  $\alpha$  satisfying  $f(\alpha) = 0$ .

By Taylor expansion show that

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2}{2} \frac{f''(\xi_n)}{f'(x_n)}$$

for some  $\xi_n \in \mathbb{R}$ . Assume that  $|f''(x)| \leq 2$  and  $|f'(x)| \geq 1$  for all real  $x$ . Prove that, if  $|x_0 - \alpha| < 1$ , then  $x_n$  converges to  $\alpha$  with order 2.

**(Problem II)**

**(14 points). Iterative Solution of Linear Equations**

This question concerns the solution of systems of linear equations in the form

$$Ax = b,$$

where  $A$  is an  $m \times m$  matrix and  $x$  and  $b$  are vectors of length  $m$ .

(a) Describe the simplest form of *iterative improvement* (also known as *residual correction* or *iterative refinement*) to solve the linear system. Describe, and briefly explain, the effect of machine precision on this algorithm.

(b) Given a matrix  $C$  which approximates the inverse of  $A$ , consider the following general residual correction method for the solution of the linear system:

$$r^m = b - Ax^m,$$

$$x^{m+1} = x^m + Cr^m.$$

State the precise condition under which this iteration converges; prove your assertion.

(c) Write the matrix  $A$  in the form  $A = L + D + U$  where  $L$  and  $U$  are (strictly) lower and upper triangular respectively and  $D$  is diagonal, define the Jacobi and Gauss-Siedel iterations for the solution of the linear system.