

SOLUTIONS:

Numerical Analysis

Solutions to Comprehensive: Numerical Analysis (30 points). Fall 1994

(Problem I)

(15 points). Linear Systems and Matrices

(a) Define the condition number \mathcal{K} of a matrix A . If

$$Ax = b, \quad Ay = b + r$$

state the upper and lower bounds relating the relative error $\frac{\|x-y\|}{\|x\|}$ to the relative perturbation $\frac{\|r\|}{\|b\|}$. Briefly discuss the implications of these inequalities for the numerical solution of linear systems.

SOLUTION: Definition is

$$\mathcal{K}(A) = \|A\| \|A^{-1}\|.$$

The basic relationship is

$$\frac{1}{\mathcal{K}(A)} \frac{\|r\|}{\|b\|} \leq \frac{\|x-y\|}{\|x\|} \leq \mathcal{K}(A) \frac{\|r\|}{\|b\|}.$$

This is important in numerical analysis, because backward error analysis shows that, instead of solving $Ax = b$ the computed solution exactly solves $Ay = b + r$ for some small vector r . The basic relationship above then shows the confidence with which we may interpret the numerical solution: if $\mathcal{K}(A)$ is large, the numerical errors may be large.

(b) Let A be an $m \times m$ symmetric matrix with eigenvalues λ_i and corresponding eigenvectors ϕ_i . What are the important properties of the λ_i and ϕ_i which follow from A being symmetric.

SOLUTION: The eigenvalues are real. The eigenvectors are orthogonal so that

$$\phi_i^T \phi_j = \delta_{ij}$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

(c) Let $\|\cdot\|$ denote the Euclidean vector norm and also use the same notation for the induced matrix norm. By noting that any vector can be expressed as a linear combination of the ϕ_i , prove that

$$\|A\| = \max_{1 \leq i \leq m} |\lambda_i|.$$

SOLUTION: Let

$$v = \sum_{j=1}^m a_j \phi_j.$$

Then

$$\|v\|^2 = \sum_{j=1}^m |a_j|^2.$$

Also

$$Av = \sum_{j=1}^m \lambda_j a_j \phi_j$$

so that

$$\|Av\|^2 = \sum_{j=1}^m \lambda_j^2 |a_j|^2.$$

Thus

$$\|Av\|^2 \leq R^2 \|v\|^2.$$

Thus

$$\|A\|^2 := \sup_{\|v\|=1} \|Av\| \leq R^2.$$

Hence $\|A\| \leq R$. To prove that $\|A\| = R$ note that $A\phi_l = \lambda_l\phi_l$ and choosing that value of l for which $|\lambda_l|$ is maximized we see that $\|A\| = R$.

(d) Calculate $\mathcal{K}(A)$ in the Euclidean norm for a symmetric positive definite matrix.

SOLUTION: We have that A^{-1} has eigenvalues λ_i^{-1} . Thus

$$\|A^{-1}\| = 1/r, \quad r := \min_i \|\lambda_i\|.$$

Hence

$$\mathcal{K}(A) = R/r.$$

(Problem II)

(10 points). Quadrature

(a) Define the composite trapezoidal rule for the approximate integration of

$$I := \int_a^b f(x) dx$$

over n intervals of equal length $h = (b-a)/n$. State the magnitude of the error in terms of h , under an assumption on the smoothness of f which you should state.

SOLUTION: The rule is

$$I \approx \frac{1}{2}[f_0 + f_n] + [f_2 + f_3 + \dots + f_n].$$

Here $f_j = f(x_j)$ and $x_j = a + jh$. The error is $\mathcal{O}(h^2)$ provided $f \in C^2([a, b], R)$.

(b) Show the weights $\{w_i\}_{i=1}^n$ and nodes $\{x_i\}_{i=1}^n$ in the Gaussian quadrature formula

$$\int_{-1}^1 f(x) dx \approx \sum_{j=1}^n w_j f(x_j)$$

satisfy

$$\sum_{j=1}^n w_j (x_j)^i = 0, \quad i = 1, 3, \dots, 2n-1$$

and

$$\sum_{j=1}^n w_j (x_j)^i = \frac{2}{i+1}, \quad i = 0, 2, \dots, 2n-2.$$

SOLUTION: Gaussian integration is chosen to approximate on all polynomials of degree $j : 0 \leq j \leq 2n-1$. This is equivalent to asking that the rule exactly integrate x^j for $j = 0, \dots, 2n-1$. Thus

$$\int_{-1}^1 x^j dx = \sum_{j=1}^n w_j x_j^j.$$

Evaluating the integral gives the result.