

Computer Science Department
Stanford University
Comprehensive Examination in Numerical Analysis
Autumn 2000

Instructions :

- Answer all questions. Total 30 points.
 - Marks for each question and its parts are specified.
 - Books and notes are not permitted.
 - Be brief and clear.
-

1. [10 marks] Solution of linear equations.

Suppose we want to solve the linear system $Ax = b$.

- (a) Give a 2×2 example where Gaussian Elimination breaks down on the first step. [2 points]
- (b) Describe the partial pivoting strategy and the complete pivoting strategy. [5]
- (c) In what situation would one prefer partial pivoting, and when would one use complete pivoting? [3]

2. [8 marks] Iterative solution of linear equations.

Let A be an $n \times n$ matrix and b a given vector. We write $A = D + L + U$, where D is the diagonal of A , and L and U are the strictly lower and upper triangular parts of A , respectively.

- (a) Define the Jacobi and Gauss-Seidel iterations for solving the system of equations. [4]
- (b) Give a sufficient condition for the approximates to converge to the solution x , independent of the initial guess. [4]

3. [12 marks] Ordinary differential equations.

Consider the equation

$$\begin{aligned}y'(x) &= \lambda y(x), \\ y(0) &= 1.\end{aligned}$$

Suppose that λ is a real negative number.

- (a) Give the exact analytic solution to the o.d.e and discuss the solution as $x \rightarrow \infty$. [2]

- (b) Suppose we use the Forward Euler method:

$$y_{k+1} = y_k + hy'_k, \quad k = 0, 1, 2, \dots$$

Under what conditions will $y_k \rightarrow 0$ as $k \rightarrow \infty$? [3]

- (c) Consider now the backward Euler method:

$$y_{k+1} = y_k + hy'_{k+1}, \quad k = 0, 1, 2, \dots$$

Under what conditions will $y_k \rightarrow 0$ as $k \rightarrow \infty$? [3]

- (d) Consider now a *general* Initial Value Problem of the form $y' = f(x, y)$, with an initial value $y(x_0) = y_0$. Denote $x_k = x_0 + kh$, and suppose that y_k is the approximation to the solution $y(x_k)$, that is obtained by applying the forward Euler method. Give an estimate for

$$|y(kh) - y_k|.$$

[4]

3. [12 marks] Ordinary differential equations.

Consider the equation

$$y'(x) = \lambda y(x),$$

$$y(0) = 1.$$

Suppose that λ is a real negative number.

- (a) Give the exact analytic solution to the o.d.e and discuss the solution as $x \rightarrow \infty$. [2]

- (b) Suppose we use the Forward Euler method:

$$y_{k+1} = y_k + hy'_k, \quad k = 0, 1, 2, \dots$$

Under what conditions will $y_k \rightarrow 0$ as $k \rightarrow \infty$? [3]

- (c) Consider now the *backward* Euler method:

$$y_{k+1} = y_k + hy'_{k+1}, \quad k = 0, 1, 2, \dots$$

Under what conditions will $y_k \rightarrow 0$ as $k \rightarrow \infty$? [3]

- (d) Consider now a *general* Initial Value Problem of the form $y' = f(x, y)$, with an initial value $y(x_0) = y_0$. Denote $x_k = x_0 + kh$, and suppose that y_k is the approximation to the solution $y(x_k)$, that is obtained by applying the forward Euler method. Give an estimate for

$$|y(kh) - y_k|.$$

[4]