

**Solutions to Comprehensive: Numerical Analysis (30 points). Fall 1995**

**(Problem I)**

**(15 points). Linear Algebra**

(a) Let  $A$  be a real  $n \times n$  symmetric matrix with  $n$  distinct real eigenvalues. Show that the eigenvectors of  $A$  are orthogonal to one another. If  $A$  is also positive definite show that the eigenvalues are positive.

SOLUTION: Let

$$Ax = \lambda x, \quad Ay = \omega y.$$

Then

$$\lambda x^T y = (Ax)^T y = x^T (A^T y) = x^T (Ay) = \omega x^T y.$$

Thus

$$(\lambda - \omega)x^T y = 0.$$

Since  $\lambda \neq \omega$  we have  $x^T y = 0$  as required. Since  $A$  is positive definite:

$$0 < x^T Ax = \lambda x^T x$$

and  $\lambda > 0$  follows.

(b) Define the Euclidean norm of a vector in  $R^n$ .

SOLUTION: If  $v = (v_1, \dots, v_n)$  then

$$\|v\|^2 := \sum_{j=1}^n v_j^2.$$

Thus  $\|v\|^2 = v^T v$ .

(c) Let  $A$  be a real, positive-definite,  $n \times n$  symmetric matrix with eigenvalues  $0 < \lambda_1 < \dots < \lambda_n$ . Let

$$\|x\|_A^2 = x^T Ax.$$

Prove that  $\|\cdot\|_A$  as defined is indeed a norm on  $R^n$ .

SOLUTION: We need to show the following three things:

- $\|x\|_A \geq 0$  and  $\|x\|_A = 0$  if and only if  $x = 0$ . This follows from the fact that  $A$  is positive definite;
- $\|\alpha x\|_A = |\alpha| \|x\|_A$  for any scalar  $\alpha$ . But

$$\|\alpha x\|_A^2 = (\alpha x)^T A(\alpha x) = \alpha^2 x^T Ax = \alpha^2 \|x\|_A^2$$

and the result follows.

- $\|x + y\|_A \leq \|x\|_A + \|y\|_A$ . Now:

$$\|x + y\|_A^2 = x^T Ax + y^T Ay + x^T Ay + y^T Ax$$

Now let  $C = \sqrt{A}$  so that  $C^2 = A$ . Then

$$\|x + y\|_A^2 = x^T Ax + y^T Ay + (Cx)^T Cy + (Cy)^T Cx$$

and by the Cauchy-Schwarz inequality

$$\|x + y\|_A^2 \leq x^T Ax + y^T Ay + 2\|Cx\| \|Cy\|.$$

But  $\|Cx\|^2 = x^T C^2 x = x^T Ax = \|x\|_A$ . Hence

$$\|x + y\|_A^2 \leq x^T Ax + y^T Ay + 2\|x\|_A \|y\|_A$$

and taking square-roots yields the required result.

(Problem II)

(15 points). Differential Equations and Quadrature

(a) Define the composite trapezoidal rule for the approximate integration of

$$I := \int_a^b f(x) dx$$

over  $n$  intervals of equal length  $h = (b - a)/n$ . State the order of accuracy of the methods in terms of  $h$ , under an assumption on the smoothness of  $f$  which you should state.

SOLUTION: The rule is

$$I \approx \frac{h}{2}[f_0 + f_n] + h[f_2 + f_3 + \dots + f_{n-1}].$$

Here  $f_j = f(x_j)$  and  $x_j = a + jh$ . The error is  $\mathcal{O}(h^2)$  provided  $f \in C^2([a, b], \mathbb{R})$ .

(b) Use the quadrature rule derived in (a) to derive a numerical approximation to the differential equation

$$\frac{du}{dt} = f(t), \quad u(\tau) = u_0,$$

at time  $t = \tau + 1$ , by partitioning the interval  $[\tau, \tau + 1]$  into  $n$  equal subintervals.

SOLUTION: The differential equation can be integrated to give

$$u(t) = u_0 + \int_{\tau}^t f(s) ds.$$

Thus

$$u(\tau + 1) = u_0 + \int_{\tau}^{\tau+1} f(s) ds.$$

We introduce the mesh points  $t_n = \tau + nh$  where  $Nh = 1$  and let  $f_n = f(t_n)$ . Applying the quadrature rule to the integral shows that

$$u(\tau + 1) \approx u_0 + \frac{1}{2}[f_0 + f_N] + [f_2 + f_3 + \dots + f_{N-1}].$$

(c) How big (in terms of  $h$ ) would you expect the error to be if you applied the method in (b) to the equation

$$\frac{du}{dt} = t^{3/2}, \quad u(\tau) = u_0,$$

with  $\tau \geq 0$ . What are the practical implications of using the method on this problem if  $\tau$  is very small?

SOLUTION If  $\tau > 0$  then  $f(t) = t^{3/2} \in C^2([\tau, \tau + 1], \mathbb{R})$  and the error will be  $\mathcal{O}(h^2)$ . If  $\tau = 0$  then  $f(t)$  is not a  $C^2$  function on the interval in question and the error will be smaller (in fact  $\mathcal{O}(h)$ .) In practice, if  $\tau$  is small, the method will require very small  $h$  before second order convergence is observed.