

Comprehensive: Numerical Analysis (30 points). Autumn 1992

(Problem I)

(16 points). **Rootfinding for Nonlinear Equations**

(a) Define the *order of convergence* of a sequence $\{x_n | n \geq 0\}$ to a point α . When is the convergence said to be *linear*? If the convergence is linear, define the *rate of linear convergence*.

(b) Given a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ and two points $a, b : f(a)f(b) < 0$ define the *bisection method*, in algorithmic form, to find a root α in $[a, b]$ satisfying $f(\alpha) = 0$.

Let $c_1 = (a + b)/2$ and let $\{c_n | n \geq 1\}$ be the sequence of approximations to α generated by the method. Show that, at each step of the iteration, c_n is the mid-point of an interval I_n in which α is guaranteed to lie and whose length L_n satisfies

$$L_n = \frac{|b - a|}{2^{n-1}}.$$

Deduce that

$$|\alpha - c_n| \leq \frac{|b - a|}{2^n}.$$

What is the rate of convergence in this case?

(c) Given a twice continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ define Newton iteration to generate a sequence $\{x_n | n \geq 0\}$ to locate a root α satisfying $f(\alpha) = 0$.

Assume that $|f''(x)| \leq 2$ and $|f'(x)| \geq 1$ for all real x . By Taylor expansion show that

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2}{2} \frac{f''(\xi_n)}{f'(x_n)}$$

for some $\xi_n \in \mathbb{R}$. Prove that, if $|x_0 - \alpha| < 1$, then x_n converges to α with order 2.

(Problem II)

(14 points). **Linear Systems and Matrices**

(a) Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 - 3x_2 &= .4 \end{aligned}$$

Write this in the form $Ax = b$ where $x = (x_1, x_2, x_3)^T$ and the ordering of the equations is retained. Show that the *LU* factorization of A does not exist.

(b) Give a permutation B of the matrix A for which the *LU* factorization does exist. Calculate this factorization.

(c) Define the condition number \mathcal{K} of a matrix A . If

$$Ax = b, \quad Ay = b + r$$

state the two basic inequalities which relate the relative error $\frac{\|x - y\|}{\|x\|}$ to the relative perturbation $\frac{\|r\|}{\|b\|}$. Briefly discuss the implications of these inequalities for the numerical solution of linear systems.

SOLUTION: The Newton algorithm is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Now

$$0 = f(\alpha) = f(x_n + \alpha - x_n) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2}f''(\xi_n)$$

for some real ξ_n . Solving for α gives

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2}{2} \frac{f''(\xi_n)}{f'(x_n)}.$$

Hence, subtracting from the Newton iteration scheme we get

$$|x_{n+1} - \alpha| = \frac{|x_n - \alpha|^2 |f''(\xi_n)|}{2 |f'(x_n)|}.$$

Now, we have that

$$|f''(x)| \leq 2, \quad \frac{1}{|f'(x)|} \leq 1$$

for all real x . Thus

$$|x_{n+1} - \alpha| \leq |x_n - \alpha|^2.$$

Thus

$$|x_n - \alpha| \leq |x_0 - \alpha|^{2^n}.$$

Quadratic convergence of x_n to α follows if $|x_0 - \alpha| < 1$.

(Problem II)

(14 points). **Linear Systems and Matrices**

(a) Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 - 3x_2 &= 4 \end{aligned}$$

Write this in the form $Ax = b$ where $x = (x_1, x_2, x_3)^T$ and the ordering of the equations is retained. Show that the LU factorization of A does not exist.

SOLUTION: We take

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & -3 & 0 \end{pmatrix}$$

and $b = (0, 1, 4)^T$.

Since

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

is a submatrix of A and is singular, the LU factorisation does not exist.