

**Solutions to Comprehensive: Numerical Analysis (30 points). Fall 1998**

**(Solution I)**

**(16 points). Rootfinding for Nonlinear Equations**

(a) Define the *order of convergence* of a sequence  $\{x_n | n \geq 0\}$  to a point  $\alpha$ . When is the convergence said to be *linear*? If the convergence is linear, define the *rate of linear convergence*.

SOLUTION: A sequence of iterates  $\{x_n | n \geq 0\}$  is said to converge with *order*  $p \geq 1$  to a point  $\alpha$  if

$$|\alpha - x_{n+1}| \leq C|\alpha - x_n|^p, \quad n \geq 0$$

for some  $C > 0$ . The convergence is *linear* if  $p = 1$ . The *rate of convergence* is then given by  $C$ .

(b) Given a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and two points  $a, b : f(a)f(b) < 0$  define the *bisection method*, in algorithmic form, to find a root  $\alpha$  in  $[a, b]$  satisfying  $f(\alpha) = 0$ .

Let  $c_1 = [a + b]/2$  and let  $\{c_n | n \geq 1\}$  be the sequence of approximations to  $\alpha$  generated by the method. Show that, for some root  $\alpha$ ,

$$|\alpha - c_n| \leq \frac{|b - a|}{2^n}.$$

What can be said about the rate of linear convergence in this case?

SOLUTION: The bisection algorithm can be written as  $Bisection(a, b, f(\bullet), \epsilon)$  :

- (1) Define  $c := (a + b)/2$ ;
- (2) If  $(b - c) \leq \epsilon$  then accept the root  $c$ ;
- (3) If  $f(b)f(c) \leq 0$  then  $a := c$ ; otherwise let  $b := c$ ;
- (4) Return to (1).

Let  $L_n$  be the length of the interval in which a root  $\alpha$  is guaranteed to lie at the  $n^{\text{th}}$  step of the algorithm. Clearly  $L_1 = |b - a|$  since  $c_1$  is the mid-point of  $[a, b]$  and we know that the root  $\alpha$  lies in  $[a, b]$ . By construction it is clear that

$$L_{n+1} = \frac{1}{2}L_n.$$

Thus

$$L_n = \frac{|b - a|}{2^{n-1}}.$$

Since  $c_n$  is the midpoint of  $I_n$  and since  $\alpha$  is guaranteed to lie in  $I_n$  we deduce that

$$|c_n - \alpha| \leq \frac{1}{2}L_n = \frac{|b - a|}{2^n}.$$

The order of convergence is hence *linear* in this case and the rate is at least  $\frac{1}{2}$ .

(c) Given a twice continuously differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  define Newton iteration to generate a sequence  $\{x_n | n \geq 0\}$  to locate a root  $\alpha$  satisfying  $f(\alpha) = 0$ .

By Taylor expansion show that

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2 f''(\xi_n)}{2 f'(x_n)}$$

for some  $\xi_n \in \mathbb{R}$ . Assume that  $|f''(x)| \leq 2$  and  $|f'(x)| \geq 1$  for all real  $x$ . Prove that, if  $|x_0 - \alpha| < 1$ , then  $x_n$  converges to  $\alpha$  with order 2.

SOLUTION: The Newton algorithm is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Now

$$0 = f(\alpha) = f(x_n + \alpha - x_n) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2}f''(\xi_n)$$

for some real  $\xi_n$ . Solving for  $\alpha$  gives

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2}{2} \frac{f''(\xi_n)}{f'(x_n)}.$$

Hence, subtracting from the Newton iteration scheme we get

$$|x_{n+1} - \alpha| = \frac{|x_n - \alpha|^2 |f''(\xi_n)|}{2 |f'(x_n)|}.$$

Now, we have that

$$|f''(x)| \leq 2, \quad \frac{1}{|f'(x)|} \leq 1$$

for all real  $x$ . Thus

$$|x_{n+1} - \alpha| \leq |x_n - \alpha|^2.$$

Thus

$$|x_n - \alpha| \leq |x_0 - \alpha|^{2^n}.$$

Quadratic convergence of  $x_n$  to  $\alpha$  follows if  $|x_0 - \alpha| < 1$ .

**(Problem II)**

(14 points). **Iterative Solution of Linear Equations**

This question concerns the solution of systems of linear equations in the form

$$Ax = b,$$

where  $A$  is an  $m \times m$  matrix and  $x$  and  $b$  are vectors of length  $m$ .

(a) Describe the simplest form of *iterative improvement* (also known as *residual correction* or *iterative refinement*) to solve the linear system. Describe, and briefly explain, the effect of machine precision on this algorithm.

SOLUTION:

Given an approximation  $x^m$  to the solution  $x$  of the linear system, the residual  $r^m$  is defined by

$$r^m = b - Ax^m.$$

Iterative improvement generates the sequence of approximations

$$x^{m+1} = x^m + \hat{e}^m$$

where  $\hat{e}$  is the computed solution of

$$A\hat{e}^m = r^m.$$

For such an iteration it is important to obtain accurate values for  $r^m$  relative to the precision used in the remainder of the calculation. The reason is simply that otherwise the errors in  $\hat{e}^m$  may be comparable with the errors in the original calculation.

(b) Given a matrix  $C$  which approximates the inverse of  $A$ , consider the following general residual correction method for the solution of the linear system:

$$\begin{aligned} r^m &= b - Ax^m, \\ x^{m+1} &= x^m + Cr^m. \end{aligned}$$

State the precise condition under which this iteration converges; prove your assertion.

SOLUTION: *The iteration converges provided that the spectral radius of the matrix  $I - CA$  is less than one.* To prove this note that the iteration gives

$$x^{m+1} - x = x^m - x + C[b - Ax^m] = x^m - x + C[Ax - Ax^m].$$

Hence the error  $\delta^m = x^m - x$  satisfies

$$\delta^{m+1} = [I - CA]\delta^m.$$

It is well-known that a necessary and sufficient condition for the existence of a matrix norm in which  $B$  is less than 1 is for the spectral radius of  $B$  to be less than one. Using this norm gives

$$\|\delta^{m+1}\| \leq \zeta \|\delta^m\|$$

for some  $\zeta \in (0, 1)$ . Iterating this gives

$$\|\delta^m\| \leq \zeta^m \|\delta^0\|$$

and hence proves convergence.

(c) Write the matrix  $A$  in the form  $A = L + D + U$  where  $L$  and  $U$  are (strictly) lower and upper triangular respectively and  $D$  is diagonal, define the Jacobi and Gauss-Siedel iterations for the solution of the linear system.

SOLUTION: We have

$$(L + D + U)x = b.$$

The *Jacobi iteration* is to generate  $x^m$  according to

$$Dx^{m+1} = b - [L + U]x^m.$$

The *Gauss-Siedel iteration* is to generate  $x^m$  according to

$$(L + D)x^{m+1} = b - Ux^m.$$