Computer Science Department
Stanford University
Comprehensive Examination in Numerical Analysis
Fall 2003

1. **Vector and Matrix Norms [8 pts]**

The following definitions hold for the norm and condition number of a *rectangular* $m \times n$ matrix A with respect to a specific matrix norm

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \qquad cond(A) = \left(\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}\right) \cdot \left(\min_{x \neq 0} \frac{\|Ax\|}{\|x\|}\right)^{-1}$$

Given the singular value decomposition $A = U\Sigma V^T$ of the matrix A (where U and V are orthogonal and Σ is the $m \times n$ diagonal matrix containing the singular values of A)

i. [3 pts] Prove that $\|A\|_2 = \|\Sigma\|_2$ using the definition above

ii. [5 pts] Prove that

$$\|A\|_2 = \sigma_{\max} \qquad \text{and} \qquad cond_2(A) = \sigma_{\max} / \sigma_{\min}$$

where $\sigma_{\max}$ is the largest singular value of A and $\sigma_{\min}$ the smallest one.

<u>Solutions</u>

i. For any $x \in R^n \setminus \{\vec{0}\}$ we have

$$\frac{\|Ax\|_2^2}{\|x\|_2^2} = \frac{(Ax)^T Ax}{x^T x} = \frac{x^T A^T Ax}{x^T x} = \frac{x^T (U\Sigma V^T)^T U\Sigma V^T x}{x^T x} = \frac{x^T V\Sigma^T U^T U\Sigma V^T x}{x^T x} = \frac{x^T V\Sigma^T \Sigma V^T x}{x^T VV^T x} =$$

$$= \frac{(\Sigma V^T x)^T \Sigma V^T x}{(V^T x)^T V^T x} = \frac{\|\Sigma V^T x\|_2^2}{\|V^T x\|_2^2} \Rightarrow \frac{\|Ax\|_2}{\|x\|_2} = \frac{\|\Sigma V^T x\|_2}{\|V^T x\|_2}$$

Since the mapping $x \mapsto V^T x$ is an isomorphism on $R^n \setminus \{\vec{0}\}$ (its inverse is simply $x \mapsto Vx$) we have

$$\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{x \neq 0} \frac{\|\Sigma V^T x\|_2}{\|V^T x\|_2} = \max_{V^T x \neq 0} \frac{\|\Sigma (V^T x)\|_2}{\|V^T x\|_2} = \max_{y \neq 0} \frac{\|\Sigma y\|_2}{\|y\|_2}$$

thus $\|A\|_2 = \|\Sigma\|_2$

ii. Let σ_i, $i = 1,\ldots,n$ be the singular values of A, forming the diagonal of the matrix Σ. Let $\sigma_k = \sigma_{\max}$ be the largest and $\sigma_l = \sigma_{\min}$ the smallest among them. Then

$$\left.\begin{aligned}
&\max_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \max_{x\neq 0}\frac{\|\Sigma x\|_2}{\|x\|_2} = \max_{x\neq 0}\sqrt{\frac{\sum_i \sigma_i^2 x_i^2}{\sum_i x_i^2}} \leq \max_{x\neq 0}\sqrt{\frac{\sum_i \sigma_{\max}^2 x_i^2}{\sum_i x_i^2}} = \sigma_{\max}\\
&\max_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \max_{x\neq 0}\frac{\|\Sigma x\|_2}{\|x\|_2} \geq \frac{\|\Sigma e_k\|_2}{\|e_k\|_2} = \frac{\|\sigma_k e_k\|_2}{1} = \sigma_k = \sigma_{\max}
\end{aligned}\right\} \Rightarrow \max_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sigma_{\max}$$

and

$$\left.\begin{aligned}
&\min_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \min_{x\neq 0}\frac{\|\Sigma x\|_2}{\|x\|_2} = \min_{x\neq 0}\sqrt{\frac{\sum_i \sigma_i^2 x_i^2}{\sum_i x_i^2}} \geq \min_{x\neq 0}\sqrt{\frac{\sum_i \sigma_{\min}^2 x_i^2}{\sum_i x_i^2}} = \sigma_{\min}\\
&\min_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \min_{x\neq 0}\frac{\|\Sigma x\|_2}{\|x\|_2} \leq \frac{\|\Sigma e_l\|_2}{\|e_l\|_2} = \frac{\|\sigma_l e_l\|_2}{1} = \sigma_l = \sigma_{\min}
\end{aligned}\right\} \Rightarrow \min_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sigma_{\min}$$

Therefore, using the definition we have $\|A\|_2 = \sigma_{\max}$ and $cond_2(A) = \sigma_{\max} / \sigma_{\min}$

2. **Differential Equations [10 pts]**

i. [4 pts] Given a square matrix A whose eigenvalues have negative real parts, show that the matrix $I - A$ is invertible and the eigenvalues of $B = (I-A)^{-1}(I+A)$ are given by the formula $\lambda_i^B = \dfrac{1+\lambda_i^A}{1-\lambda_i^A}$, where λ_i^A are the eigenvalues of A

ii. [6 pts] Consider the vector ordinary differential equation $\vec{y}' = f(x, \vec{y})$ and the implicit trapezoidal method for solving it:

$$\vec{y}_{k+1} = \vec{y}_k + h\frac{f(x_k, \vec{y}_k) + f(x_{k+1}, \vec{y}_{k+1})}{2}$$

Prove that this method is unconditionally stable when applied to the model vector ODE $\vec{y}' = A\vec{y}$ for a matrix A whose eigenvalues have negative real parts (that is, show that $\|\vec{y}_k\| \to 0$ as $k \to \infty$, regardless of the value of the step size h)