

Solution to the Numerical Analysis Examination

I.

a) The LU factorization does not exist since the 2×2 submatrix

$$A_2 = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

is singular. A necessary and sufficient condition is that all the submatrices $A_k = (a_{ij})_{1 \leq i, j \leq k}$, $k = 1, \dots, n$ are nonsingular.

b) A permutation of the second and third row gives

$$B = \begin{bmatrix} 4 & -2 & 1 \\ 3 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix} \quad \text{with the } LU \text{ factorization}$$

$$B = \begin{bmatrix} 1 & & & \\ 3/4 & 1 & & \\ -1/2 & 0 & 1 & \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 9/2 & 1/4 \\ 0 & 0 & 1/2 \end{bmatrix}$$

c) The third column of A^{-1} can be computed by solving the linear system $Ax = e_3$ which is equivalent to $Bx = e_2$. This is done by solving the triangular system $Ly = e_2$, and then $Ux = y$.

d)

$$\begin{aligned} a - 2b + 4c &= -1 \\ a - b + c &= 1 \\ a &= 1 \\ a + b + c &= 4 \end{aligned}$$

The corresponding linear system is

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

Since $a = 1$, we have

$$\begin{aligned} -2b + 4c &= -2 \\ -b + c &= 0 \\ b + c &= 3 \end{aligned}$$

The second and third equation lead to $b = c = 3/2$ but then the first equation is not fulfilled ($-2 \cdot 3/2 + 4 \cdot 3/2 \neq -2$!).

II.

a)

$$\begin{aligned} a_0 &= 1 \\ a_0 + \frac{1}{2}a_1 + \frac{1}{4}a_2 &= \frac{2}{3} \\ a_0 + a_1 + a_2 &= \frac{1}{2} \end{aligned}$$

$a_0 = 1$:

$$\left. \begin{aligned} \frac{1}{2}a_1 + \frac{1}{4}a_2 &= -\frac{1}{3} \\ a_1 + a_2 &= -\frac{1}{2} \end{aligned} \right\} \Rightarrow \begin{aligned} a_1 &= -\frac{5}{6} \\ a_2 &= \frac{1}{3} \end{aligned}$$

The interpolating polynomial is $p_2(x) = \frac{1}{3}x^2 - \frac{5}{6}x + 1$.

b)

$$|f(x) - p_2(x)| \leq \frac{1}{6} \cdot \max_{x \in [0,1]} |f^{(3)}(x)| \cdot \max_{x \in [0,1]} |w_2(x)|$$

$$f^{(3)}(x) = -\frac{1}{(1+x)^4} \Rightarrow \max_{x \in [0,1]} |f^{(3)}(x)| \leq 1$$

$$w_2(x) = x(x - \frac{1}{2})(x - 1), \quad w_2'(x) = 3x^2 - 3x + \frac{1}{2}$$

$$\max_{x \in [0,1]} |w_2(x)| = |w_2(\frac{1}{2} \pm \sqrt{\frac{1}{12}})| = \frac{1}{6} \sqrt{\frac{1}{12}}$$

c) Suppose p_n and \tilde{p}_n are two polynomials interpolating $f(x)$ at the points $x_0 < x_1 < \dots < x_n$. Then, $\hat{p}_n = p_n - \tilde{p}_n$ is a polynomial of degree n with $n+1$ zeros. Therefore, $\hat{p}_n \equiv 0$ and $p_n \equiv \tilde{p}_n$.