Numerical Analysis Comprehensive Exam

Fall 2007

Use the attached sheets to write your answers.

1. Let $A \in \mathbb{R}^{m \times n}$ where $m \geq n$, and let $b \in \mathbb{R}^m$. Consider the least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 \tag{1}$$

   (a) (3 points) Let $\phi(x) = \frac{1}{2}\|Ax - b\|_2^2$. A minimizer of $\phi(x)$, $x \in \mathbb{R}^n$ satisfies the equation $\nabla \phi(x) = 0$. Use this to derive a linear equation whose solution solves (1). How would you solve such a system?

   (b) (2 points) Under what conditions is a solution of (1) unique?

   (c) Let $A = U \Sigma V^T$ be the singular value decomposition of A, and let $\text{rank}(A) = r$. Let U and V be written in terms of their columns as

   $$U = [u_1, \ldots, u_m], \quad V = [v_1, \ldots, v_n]$$

   Let

   $$\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r, \underbrace{0, \ldots, 0}_{n-r})$$

   i. (4 points) The minimum norm solution x_{LS} is the solution of (1) such that $\|x_{LS}\|_2 \leq \|x\|_2$ for any solution x. Show that the minimum norm solution of (1) is given by

   $$x_{LS} = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i$$

   ii. (2 points) Show that the associated residual has norm

   $$\|Ax_{LS} - b\|_2^2 = \sum_{i=r+1}^{m} (u_i^T b_i)^2$$

2. Consider the scalar initial value problem $\dot{y} = \lambda y$, $x \in \mathbb{R}$, $y(x_0) = y_0$, $\lambda < 0$.

   (a) (1 point) Give the update rule for the numerical solution using backward Euler.

   (b) (1 point) Give the update rule for the numerical solution using trapezoidal rule.

   (c) (2 points) Describe the qualitative behavior of both update rules as Δt becomes very large.

3. Consider the equation $m\ddot{x} + c\dot{x} + kx = 0$ where $m > 0$, $c > 0$, $k > 0$, $c^2 > 4km$.

   (a) (2 points) Convert the equation to a system of the form $\dot{u} = Au$ and find the eigenvalues of the matrix.

   (b) (2 points) What choice of Δt will make forward Euler absolutely stable?

   (c) (2 points) What choice of Δt will make trapezoidal rule absolutely stable?

4. (4 points) Describe an algorithm for finding $z \in \mathbb{R}^n$, $z \neq 0$ such that $Uz = 0$ where $U \in \mathbb{R}^{n \times n}$ is upper triangular and $u_{nn} = 0$, $u_{11}, \ldots, u_{n-1,n-1} \neq 0$.

5. Consider vectors $x, y \in \mathbb{R}^n$ such that $\|x\|_2 = \|y\|_2 \neq 0$.

   (a) (3 points) Find a reflection matrix A such that $Ax = y$.

   (b) (2 points) Describe how you would compute Az for a given vector z in $O(n)$ time.

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