2007 Comprehensive Examination
Logic

Honor code and all that stuff.

1. Logical Entailment. (20 points) Let Γ and Δ be sets of closed sentences in first-order logic, and let φ and ψ be individual closed sentences in first-order logic. State whether each of the following statements is true or false. No explanation is necessary.

(a) If $\Gamma \cap \Delta \models \varphi$, then $\Gamma \models \varphi$ and $\Delta \models \varphi$.

(b) If $\Gamma \cup \Delta \models \varphi$, then $\Gamma \models \varphi$ or $\Delta \models \varphi$.

(c) $\Delta \models (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \models \psi$.

(d) $\Delta \models \varphi$ or $\Delta \models \psi$ if and only if $\Delta \models (\varphi \vee \psi)$.

(e) If $\Delta \models \varphi$ and $\Delta \models \neg\psi$, then $\Delta \not\models (\varphi \Rightarrow \psi)$.

(f) If $\Delta \models \varphi$ and $\Delta \models \psi$, then $\Delta \models (\varphi \Rightarrow \psi)$.

(g) If $\Delta \models p(\tau)$ for some ground term τ, then $\Delta \not\models \forall x.\neg p(x)$.

(h) If $\Delta \models p(\tau)$ for every ground term τ, then $\Delta \models \forall x.p(x)$.

(i) If $\Delta \models \forall x.(p(x) \Rightarrow q(x))$, then $\Delta \models \exists x.(p(x) \wedge q(x))$.

(j) If $\Gamma \models (\varphi \Rightarrow \psi)$ and $\Delta \models (\psi \Rightarrow \varphi)$,then $\Gamma \cap \Delta \models (\varphi \Rightarrow \psi) \vee (\psi \Rightarrow \varphi)$.

2. Unification. (10 points)

(a) Assuming that x, y, z, v, w are variables, give a most general unifier for the expressions $p(t(x, y), r(z, z))$ and $p(t(t(w, z), v), w)$.

(b) What is the result of applying this unifier to these expressions.

(c) Is it possible for two expressions to have more than one most general unifier? If so, give a simple example. If not, give a one-sentence explanation.

3. Clausal Form. (10 points) Convert the following sentence to clausal form.

$$(\exists z.\forall y.p(z,y) \vee \forall y.\exists z.p(z,y)) \Rightarrow \exists z.\forall y.p(y,z)$$

5. Resolution. (20 points) Use resolution to show that the following set of clauses is unsatisfiable. Assume that *w*, *x*, *y*, and *z* are variables and *a* is an object constant.

$$\{\neg p(x, y), q(x, y, f(x,y))\}$$
$$\{\neg r(y, z), q(a, y, z)\}$$
$$\{r(y, z), \neg q(a, y, z)\}$$
$$\{p(x, g(x)), q(x, g(x), z)\}$$
$$\{\neg r(x,y), \neg q(x, w, z)\}$$

Note that this is a question about Resolution Theorem Proving. You will get zero points (nil, nada, rien, zip, nothing) unless you use resolution and/or factoring on each step.

5. Model Building. (10 points) Consider the following sentence.

$$\forall x.(\neg p(x) \vee q(x)) \Leftrightarrow \neg\forall x.(p(x) \wedge q(x)).$$

(a) Give an interpretation that satisfies this sentence,

(b) Is the sentence valid? If so, write “valid”. If not, give an interpretation that falsifies it.

In your interpretation(s), use {john, paul, mary} as the universe of discourse.

6. Herbrand Models. (10 points) One popular version of the Herbrand Theorem states that a set of equality-free clauses is satisfiable if and only if it has a Herbrand model. If the word "clauses" is changed to "first-order sentences", does the theorem still hold? If so, explain why. If not, give a counterexample.

7. Theory Completeness. (20 points) A universal language is a first-order language without functions, explicit quantifiers, or equality. Free variables are universally quantified. For example, $p(x) \Rightarrow p(x)$ is equivalent to $\forall \text{x}.(p(x) \Rightarrow p(x))$ Now, consider a universal language with just one unary relation constant p and two object constants a and b. Which of the following sentences logically entail a theory that is complete for all sentences in this language? For each case, write "complete" or "incomplete".

(a) $p(a) \wedge p(b)$

(b) $p(a) \wedge \neg p(b)$

(c) $(p(a) \vee p(b)) \wedge (p(a) \vee \neg p(b)) \wedge (\neg p(a) \vee p(b)) \wedge (\neg p(a) \vee \neg p(b))$

(d) $p(x)$

(d) $p(x) \Rightarrow p(x)$

(e) $p(x) \Rightarrow \neg p(x)$

(f) $p(a) \wedge \neg p(a)$

(g) Where completeness is concerned, sentence (b) has an interesting property that differentiates from the other sentences. What is that property?