Comprehensive Exam in Logic
November 2006

37 questions in 6 parts

Time: 1 hour

Instructions

- **Do not open the test booklet until instructed to do so.**
- The exam is open book and open notes, but no laptops or electronic accessories are allowed.
- Answer each question in the exam itself. The answers will fit in the given space. Writing outside of the given space will **not** be considered.
- It is strongly recommended that you work out the answer on scratch paper before answering.
- Unanswered questions are not penalized. Incorrect answers are penalized. Read the instructions before each section carefully.
- **THE HONOR CODE:**
  1. The honor code is an undertaking of the students individually and collectively:
     (a) that they will not give or receive aid in examinations; they will not give or receive unauthorized aid in class work, in the preparation of reports, or in any other work that is to be used by the instructors as the basis of grading;
     (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the honor code.
  2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will avoid, as far as possible, academic procedures that create temptation to violate the Honor code.
  3. While the faculty alone have the right and the obligation to set academic requirements, the students and the faculty will work together to establish optimal conditions for honorable academic work.

  By writing my "magic–number" below, I acknowledge and accept the honor code.

WRITE MAGIC NUMBER: ______________________

I. Choose one answer to each of the following questions. F is a sentence (closed formula).
No answer: **0** *Incorrect answer*: **-1** *Correct answer*: **2**

1. If F is valid, then $\neg F$ is
   (*a*) satisfiable
   (*b*) unsatisfiable
   (*c*) valid
   (*d*) none of the above

2. If F is T-valid for some theory T, then F is necessarily __________ in FOL.
   (*a*) satisfiable
   (*b*) unsatisfiable
   (*c*) valid
   (*d*) none of the above

3. If F is T-satisfiable for some theory T, then F is necessarily __________ in FOL.
   (*a*) satisfiable
   (*b*) unsatisfiable
   (*c*) valid
   (*d*) none of the above

4. If F is T_1-valid and T_2-valid, then F is necessarily $(T_1 \cup T_2)$-valid.
   (*a*) True
   (*b*) False

5. If F is T-valid for complete theory T, then $\neg F$ is necessarily T-__________.
   (*a*) satisfiable
   (*b*) unsatisfiable
   (*c*) valid
   (*d*) none of the above

6. $\forall x.\ (3x = 2 \ \rightarrow\ x \leq 0)$ is __________ in the theory of integers, $T_{\mathbb{Z}}$, and __________ in the theory of rationals, $T_{\mathbb{Q}}$.
   (*a*) valid, valid
   (*b*) invalid, valid
   (*c*) valid, invalid
   (*d*) invalid, invalid

7. If $\neg F$ has a T-interpretation, then F is necessarily T-__________.
   (*a*) satisfiable
   (*b*) unsatisfiable
   (*c*) valid
   (*d*) none of the above

8. An inconsistent theory can be made consistent by adding more axioms.
   (*a*) True
   (*b*) False

II. Write a check next to the valid propositional logic formulae. Write a line through the invalid formulae.
No answer: **0** *Incorrect answer*: **-1** *Correct answer*: **1**

1. $(P \land Q) \rightarrow (P \rightarrow Q)$
2. $(P \rightarrow Q) \lor (P \land \neg Q)$
3. $(P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow R)$
4. $(P \rightarrow (Q \lor R)) \rightarrow (P \rightarrow R)$
5. $\neg(P \land Q) \rightarrow (R \rightarrow (\neg R \rightarrow Q))$
6. $(P \land Q) \lor \neg P \lor (\neg Q \rightarrow \neg P)$
7. $(P \rightarrow (Q \rightarrow R)) \rightarrow (\neg R \rightarrow (\neg Q \rightarrow \neg P))$
8. $(\neg R \rightarrow (\neg Q \rightarrow \neg P)) \rightarrow (P \rightarrow (Q \rightarrow R))$

III. Write a check next to the valid first-order logic formulae. Write a line through the invalid formulae.
No answer: **0** *Incorrect answer*: **-1** *Correct answer*: **1**

1. $(\forall x, y.\ (p(x, y) \rightarrow p(y, x))) \rightarrow \forall z.\ p(z, z)$
2. $\forall x, y.\ (p(x, y) \rightarrow (p(y, x) \rightarrow \forall z.\ p(z, z)))$
3. $p(a) \rightarrow \exists x.\ p(x)$
4. $(\exists x.\ p(x)) \rightarrow \forall y.\ p(y)$
5. $(\forall x.\ p(x)) \rightarrow \exists y.\ p(y)$

IV. Write a check next to the decidable first-order theories and fragments. Write a line through the undecidable ones. *QFF* abbreviates *quantifier-free fragment*.
No answer: **0** *Incorrect answer*: **-1** *Correct answer*: **1**

1. Propositional logic
2. Empty theory (all of first-order logic)
3. Theory of equality
4. QFF of theory of equality
5. First-order Peano arithmetic
6. QFF of first-order Peano arithmetic
7. Presburger arithmetic
8. QFF of Presburger arithmetic
9. Theory of reals with addition and multiplication (elementary algebra)
10. QFF of theory of elementary algebra
11. Theory of (possibly cyclic) LISP-like lists
12. QFF of theory of LISP-like lists
13. Theory of arrays
14. QFF of theory of arrays

V. Find a most general unifier for the following pair of terms, if one exists; otherwise, show where unification fails.
Incorrect or no answer: 0 *Correct answer*: 10 *Partial credit may be given.*

$$\langle p(g(y), x, f(g(y))),\ p(z, h(z, w), f(w))\rangle$$

VI. Annotate the following function so that its annotations are inductive. The variable rv in the function postcondition holds the *return value*.
Incorrect or no answer: 0 *Correct answer*: 10 *Partial credit may be given.*

```
@pre ⊤
@post ∀i. 0 ≤ i < |rv| → rv[i] ≥ 0
int[] abs(int[] a₀) {
  int[] a := a₀;
  for
    @[                                        ]
    (int i := 0; i < |a|; i := i + 1) {
    if (a[i] < 0) {
      a[i] := − a[i];
    }
  }
  return a;
}
```