Computer Science Department
Stanford University
Comprehensive Examination in Numerical Analysis
Fall 2005

1. **Solution of polynomial equations [20pts]**

Let $p_n(x)$ be a polynomial of degree n with real coefficients. Assume the roots are distinct.

(a) [5pts] Define Newton's Method for finding the roots of $p_n(x)$.

(b) [7pts] Show the rate of convergence for finding the roots.

(c) [8pts] Suppose $p_n(x) = (x-a)^k q_{n-k}(x)$, where $q_{n-k}(x)$ is a polynomial of degree $n-k$ and assume all its roots are distinct. Show how to modify Newton's Method so that the root a is obtained and convergence is quadratic.

2. **Solution of linear equations [20pts]**

(a) [7pts] Describe Gaussian elimination as a matrix factorization. What can be said about the magnitude of the elements of the factorization when partial pivoting is implemented?

(b) [8pts] Assume we have two methods for implementing Gaussian elimination so that we have

$$A = L_1 U_1 \text{ and } A = L_2 U_2.$$

Show that $L_1 = L_2$ and $U_1 = U_2$.

(c) [5pts] Suppose, we know

$$A = LU \text{ and } B = AD$$

where D is a diagonal matrix with non-zero diagonal elements. What is the relation between the LU factorization of A and that of B?

3. **Differential equations [20pts]**

Consider the differential equation

$$(*) \quad y' = f(x, y)$$
$$y(a) = \alpha.$$

There are two methods that are frequently used for developing methods for solving differential equations. They are as follows:

i.) Formulas based on quadrature.

ii.) Methods based on Taylor Series.

(a) [13pts] Develop Euler's Method from each of these two methods for solving the differential equation given by $(*)$.

(b) [7pts] Discuss the error: $|y_n - y(x_n)|$ where y_n is the numerical solution to the difference equation and $y(x_n)$ is the exact solution at x_n. Show how it depends on the mesh width: h.