SOLUTIONS Comp. Exam. in Logic
November 2005

39 questions

Time: 1 hour

Instructions

- **Do not open the test booklet until instructed to do so.**
- The exam is open book and open notes, but no laptops or electronic accessories are allowed.
- Answer each question in the booklet itself. The answers should fit the space given. Writing on the margin/footer will **not** be considered.
- It is strongly recommended that you work out the answer outside the test booklet before answering.
- All questions have penalties for wrong answers. Read the instructions carefully before you start.
- **THE HONOR CODE:**
  1. The honor code is an undertaking of the students individually and collectively:
     (a) that they will not give or receive aid in examinations; they will not give or receive unauthorized aid in class work, in the preparation of reports, or in any other work that is to be used by the instructors as the basis of grading;
     (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the honor code.
  2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will avoid, as far as possible, academic procedures that create temptation to violate the Honor code.
  3. While the faculty alone have the right and the obligation to set academic requirements, the students and the faculty will work together to establish optimal conditions for honorable academic work.

  By writing my "magic–number" below, I acknowledge and accept the honor code.

WRITE MAGIC NUMBER: ______

Some conventions: We assume a few pieces of notation: a first-order logic sentence has no free variables, while a first-order formula may have some free variables (i.e., sentences are formulas with no free variables).

Given a set of first-order sentences Σ, we denote by $\mathcal{M}(\Sigma)$ the class of models of all the sentences in Σ. Given a set of structures (also called interpretations) M, we use $Th(M)$ to denote the set of sentences satisfied by all structures in M. Given a set of sentences Σ the set of consequences of Σ, $Th(\mathcal{M}(\Sigma))$ is denoted by $Cn(\Sigma)$.

A theory is a set of sentences. A theory $\mathcal{T}$ is called axiomatizable if for some recursively enumerable set of sentences (called axioms) $Cn(\Sigma) = \mathcal{T}$.

Grading: For both sections A and B:

correct answer	2 points each
no answer	0 points each
first 5 incorrect answers (in A and B combined)	0 points each
second 5 incorrect answers (in A and B combined)	-2 points
incorrect answers beyond 10 (in A and B combined)	-4 points

Section A
For each question in this section you need to choose one out the four possible choices provided. Indicate your answer by writing your choice clearly in the box provided. Remember: **No points deducted for leaving questions unanswered**.

1. Which of the following is a propositional tautology?

   (a) $((P \to Q) \to P) \to P$.

   (b) $P \to (P \to (P \to Q))$.

   (c) $(P \to (Q \vee R)) \leftrightarrow \neg((Q \wedge R) \to P)$.

   (d) None of the above.

   a

2. Which of the following is true about complete sets of propositional connectives?

   (a) The set $\{\to, \neg\}$ is **not** a complete set of connectives.

   (b) There is some binary propositional connective (expressed for example as a truth table) that is complete by itself.

   (c) the constant **false** or **true** must always be present in a complete set of connectives.

   (d) The set $\{\wedge, \vee\}$ is a complete set of connectives.

   b

3. Let $\mathcal{F}, \mathcal{G}$ stand for arbitrary sentences of propositional logic. What is the relationship between these three statements?

   I $\mathcal{F}$ is equivalent to $\mathcal{G}$.

   II $(\mathcal{F} \equiv \mathcal{G})$ is valid.

   III $\mathcal{F}$ is valid precisely when $\mathcal{G}$ is valid.

   (a) (I), (II) and (III) are equivalent.

   (b) (I), (II) and (III) are not all equivalent, but (I) implies (II), (II) implies (III).

   (c) (I), (II) and (III) are not all equivalent, but (II) implies (III), (III) implies (I).

   (d) none of the above.

   b

4. Given the sentence $\sigma\ :\ \exists x\,[p(x)\ \wedge \neg\, p(x)]$, which of the following is correct?

   (a) σ is valid in first-order logic.

   (b) σ is valid in first-order logic, but no tableau proof can be found.

   (c) σ is not valid in any theory.

   (d) σ is not valid in first-order logic, but it is valid in some axiomatic theory.

   d

5. Recall that a well-founded relation R over a domain D is a binary relation $R \subseteq D \times D$, such that no infinite chain $a_1, a_2, \ldots$ of elements exists with $(a_i, a_{i+1}) \in R$.

   Let R be a well-founded relation and let R^{-1} be its inverse relation (i.e., $R^{-1} = \{(a, b) \mid (b, a) \in R\}$.) Which of the following is true?

   (a) R^{-1} is always a well-founded relation.

   (b) if the domain is finite, then R^{-1} is a well-founded relation.

   (c) if the domain is not finite, then R^{-1} cannot be a well-founded relation.

   (d) R^{-1} is a well-founded relation iff R is a finite set.

   b

6. Let $\mathcal{T}$ be a theory that is axiomatizable and complete. Which of the following is necessarily true?

   (a) $\mathcal{T}$ is decidable.

   (b) $\mathcal{T}$ is decidable only if it has a finite axiomatization.

   (c) $\mathcal{T}$ is decidable only if it is consistent.

   (d) $\mathcal{T}$ is decidable only if it is inconsistent.

   a

7. Let $\mathcal{N}$ be the structure of the natural numbers with 0, addition, successor, multiplication, and less-than relation. Which of the following is true?

(a) The theory $Th(\mathcal{N})$ has a finite model.

(b) The theory $Th(\mathcal{N})$ has a model with uncountable cardinality.

(c) The theory $Th(\mathcal{N})$ has a finite axiomatization.

(d) all of the above.

b

8. Let t and s be two terms, and s_1 be a proper subterm of s (i.e., s_1 is a subterm of s different from s). Which of the following is true?

(a) s can be unified with s_1.

(b) if t can be unified with s, then t cannot be unified with s_1.

(c) if t can be unified with s_1, then t cannot be unified with s.

(d) t may be unifiable with both s and s_1.

d

9. Let P be an arbitrary unary predicate. Consider a formula φ and the following first-order logic sentence σ:

$$\exists * [(\neg P(x) \vee \varphi) \rightarrow (\neg\varphi \wedge P(x))]$$

Note that for each choice of formula φ, a different sentence σ is generated.

Which of the following is necessarily true?

(a) for some φ, the sentence σ is valid.

(b) for some φ, the sentence σ is satisfiable, while for some others σ is unsatisfiable.

(c) for all φ, the sentence σ is satisfiable.

(d) for all φ, the sentence σ is unsatisfiable.

b

Section B

For each question in this section you need to circle "T" if you think the statement holds or "F" if you think it does not. Remember: **There is no penalty for leaving questions unanswered.**

1. Consider an *arbitrary* propositional logic sentence $\mathcal{F}$. If $\mathcal{F}$ is not valid, then we can prove that $(\neg\, \mathcal{F})$ is valid using the deductive tableau method.

	F

2. Consider arbitrary closed sentences $\mathcal{F}$ and $\mathcal{G}$ in first-order logic, and a set Σ of axioms. If $\mathcal{F}$ is valid in the theory of Σ (i.e., $\mathcal{F}$ is in $Cn(\Sigma)$), and $(\neg\mathcal{F})$ is valid in the theory of Σ, then $\mathcal{G}$ is valid in the theory of Σ.

T	

3. If two first-order sentences are valid then they are equivalent.

T	

4. If two sentences are equivalent in a theory $\mathcal{T}$, then they are also equivalent in the theory consisting of the Cn of all the first-order sentences not in $\mathcal{T}$.

	F

5. If the resolution rule is applied with a unifier θ that is not a most general unifier (m.g.u.) then soundness is compromised (i.e., some non-valid formula can be proved valid using a deductive tableau).

	F

6. If a theory $\mathcal{T}$ is finitely axiomatizable, then there is some sentence σ such that, if $\varphi \in \mathcal{T}$, then $\sigma \models \varphi$.

T	

7. A (not necessarily finite) set Σ of sentences has a model iff **every** finite subset $\Sigma_0 \subseteq \Sigma$ has a model.

T	

8. Let Γ be an arbitrary set of first-order sentences, and ψ be a first-order sentence. Then either $\Gamma \models \psi$ **or** $\Gamma \models \neg\psi$ (or both).

	F

9. Let ψ be an arbitrary first-order sentence. There is some set of sentences Γ for which $\Gamma \models \psi$ **and** $\Gamma \models \neg\psi$.

T	

10. The finite union of well-founded relations is a well-founded relation.

	F

11. The intersection of well-founded relations is a well-founded relation.

T	

12. The composition of well-founded relations is a well-founded relation.

	F

13. Consider the first-order language consisting of equality and a binary predicate symbol E. You can think of it as the language of directed graphs.

    The following predicate R defines reachability.

    (reachability means that for every interpretation $\mathcal{I}$ and two elements in its domain $a, b \in |\mathcal{I}|$ if $R^{\mathcal{I}}(a, b)$ then there are some $a_1, \ldots, a_n$ with $E^{\mathcal{I}}(a_i, a_{i+1})$ and $a_1 = a$, $a_n = b$.)

$$\forall x, y\, [R(x, y) \equiv \Big(E(x, y) \vee \exists z\, (R(x, z) \wedge E(z, y))\Big)]$$

	F

14. If a sentence is valid then it must always occur with positive polarity in any enclosing sentence.

	F

15. Let $\mathcal{F}$ be a sentence that is not valid, which contains a subsentence $\mathcal{G}$ with at least one occurrence with negative polarity. Replacing **all** occurrences of $\mathcal{G}$ in $\mathcal{F}$ by **true** can not generate a valid sentence.

	F

16. The set of non-valid sentences of first-order logic is **not** recursively enumerable.

T	

17. Let $\mathcal{T}$ be a decidable theory. If $\mathcal{T}$ is consistent, then it is complete.

	F

18. Let $\mathcal{T}$ be a decidable and consistent theory and Σ an axiomatization of $\mathcal{T}$. There is a way to extend Σ to a complete and still consistent theory by adding one of ψ or $\neg\psi$ for all sentences for which $\Sigma \not\models \psi$ and $\Sigma \not\models \neg\psi$.

T	

19. Let Γ be a theory. If for all models $\mathcal{I}$ of Γ either $\models_{\mathcal{I}} \varphi$ or $\models_{\mathcal{I}} \neg\varphi$, then Γ is complete.

	F

20. Let φ be a sentence over a first-order language, and $\mathcal{T}_1$ and $\mathcal{T}_2$ be two theories over the same language. If $\mathcal{T}_1 \models \varphi$ and $\mathcal{T}_2 \models \varphi$ then certainly $\mathcal{T}_1 \cup \mathcal{T}_2 \models \varphi$.

T	

21. Let φ be a sentence over a first-order language and $\mathcal{T}_1$ and $\mathcal{T}_2$ be two theories over the same language. If $\mathcal{T}_1 \not\models \varphi$ and $\mathcal{T}_2 \not\models \varphi$ then certainly $\mathcal{T}_1 \cup \mathcal{T}_2 \not\models \varphi$.

	F

22. If a theory $\mathcal{T}$ is finitely axiomatizable, then for every positive number k there is an axiomatization of $\mathcal{T}$ with exactly k sentences.

T	

23. Let $\mathcal{F}$ be a sentence with quantifiers. The following holds *if and only if* $\mathcal{F}$ is valid: *Validity-preserving skolemization transforms $\mathcal{F}$ into a sentence without quantifiers $\mathcal{G}$ that is equivalent to $\mathcal{F}$.*

	F

24. The validity preserving skolemization of

$$\exists x\, \forall z\, \exists y\, \big(p(x, y) \wedge \neg p(y, f(z))\big)$$

is **equivalent** to

$$\big(p(x', y') \wedge \neg p(y', f(f(x')))\big).$$

	F

25. Is the following a first-order validity?

$$\big([\forall x\, p(x) \rightarrow \exists x\, (Q(x) \equiv \neg Q(f(f(x))))] \rightarrow \forall y\, p(y)\big) \rightarrow \forall z\, p(z)$$

T	

26. If a propositional deductive tableau rule is sound, then the generated sub-tableau must be valid.

	F

27. If $\mathcal{F}[P]$ is unsatisfiable then $\mathcal{F}[\neg P]$ is satisfiable.

	F

28. Let a and b be constants, and x, y and z be variables. The tuple

$$\langle f(z, g(z, a)), f(x, g(b, y)), f(b, g(b, x))\rangle$$

is **not** unifiable.

T	

29. If $\forall * (\mathcal{F} \vee \mathcal{G})$ is valid, then either $\exists * \mathcal{F}$ is valid or $\exists * \mathcal{G}$ is valid (or both).

	F

30. Let $\mathcal{F}$ be valid if and only if $\mathcal{G}$ is valid, and let $\mathcal{H}'$ be obtained from $\mathcal{H}$ by replacing all occurrences of $\mathcal{F}$ by $\mathcal{G}$ in $\mathcal{H}$. Then $\mathcal{H}$ is valid **iff** $\mathcal{H}'$ is.

	F