Computer Science Department
Stanford University

Comprehensive Examination in Numerical Analysis
Fall 2004

Note: You may use a result you are asked to prove in subsequent questions even if you have not been able to prove it.

1. Norms and orthogonality [10pts]

1. [4pts] Let $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ be an orthonormal basis for R^n and $\mathbf{A}$ a $n \times n$ matrix. If $\mathbf{Aq}_1, \mathbf{Aq}_2, \ldots, \mathbf{Aq}_n$ is an orthonormal set as well, prove that $\mathbf{A}$ must be orthogonal.

2. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ nonsingular matrices satisfying $\|\mathbf{Ax}\|_2 = \|\mathbf{Bx}\|_2$ for every $\mathbf{x} \in R^n$.

   (a) [3pts] Show that $\mathbf{A}$ and $\mathbf{B}$ have the same singular values.
   (*Hint:* Show that $\mathbf{A}^T\mathbf{A} = \mathbf{B}^T\mathbf{B}$)

   (b) [3pts] Show that $\mathbf{A} = \mathbf{QB}$ for an orthogonal matrix $\mathbf{Q}$.

2. Optimization and least squares [10 pts]

If $\mathbf{A}$ is an $m \times n$, $m > n$ matrix with full column rank and $\mathbf{b} \in R^m$, we know that the least squares solution to the overdetermined system $\mathbf{Ax} = \mathbf{b}$ is given by the system of normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x}_0 = \mathbf{A}^T\mathbf{b}$ and corresponds to the vector $\mathbf{x}_0 \in R^n$ that minimizes $\phi(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2$.

1. [4pts] Consider the modified functional

$$\hat{\phi}(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2 + \mathbf{x}^T\mathbf{Bx} - 2\mathbf{c}^T\mathbf{x}$$

   where $\mathbf{B}$ is an $n \times n$ symmetric and positive definite matrix, and $\mathbf{c} \in R^n$. Show that we can find the value of $\mathbf{x}$ that minimizes $\hat{\phi}(\mathbf{x})$ by solving the following modification of the system of normal equations

$$\left(\mathbf{A}^T\mathbf{A} + \mathbf{B}\right)\mathbf{x}_0 = \mathbf{A}^T\mathbf{b} + \mathbf{c}$$

2. [1pt] Explain why we can find a symmetric, positive definite matrix $\mathbf{M}$ such that $\mathbf{B} = \mathbf{M}^2 = \mathbf{M}^T\mathbf{M}$.

3. [4pts] Show that the value of $\mathbf{x}$ that minimizes $\hat{\phi}(\mathbf{x})$ can alternatively be found by finding the least squares solution of the following modification of the original overdetermined system

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{M} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{b} \\ \mathbf{M}^{-1}c \end{pmatrix}$$

where $\mathbf{M}$ is the matrix defined in (3)

4. [1pts] Why would you prefer either of the approaches described in (1) or (3) to solve the given minimization problem?

3. Differential equations [10pts]

Consider the scalar ordinary differential equation $y' = \lambda y,\ \lambda \in R$ and the following methods for solving it

$$\begin{aligned} \text{Forward Euler} &: \quad y_{k+1} = y_k + h y'_k \\ \text{Backward Euler} &: \quad y_{k+1} = y_k + h y'_{k+1} \\ \text{Trapezoidal} &: \quad y_{k+1} = y_k + \frac{h}{2}(y'_k + y'_{k+1}) \end{aligned}$$

1. [4pts] Prove that taking one step of forward Euler to get from y_k to y_{k+1} followed by a step of backward Euler to get from y_{k+1} to y_{k+2} is equivalent to taking one trapezoidal step from y_k to y_{k+2} (note that the integration interval will be equal to $2h$ for a step from y_k to y_{k+2}).

2. [4pts] Prove that we get the same result if we use backward Euler for the first step and forward Euler for the second.

3. [2pts] Explain why both methods described in (1) and (2) are stable for any $\lambda < 0$