Stanford University
Computer Science Department

Fall 2004 Comprehensive Exam in Numerical Analysis

1. **Open book & notes/ No Laptops. Write only in the Blue Book.**
2. **The exam is timed for 60 minutes.**
3. **Write your Magic Number on this cover sheet and on the Blue Book.**

The following is a statement of the Stanford University Honor Code:

A. The Honor Code is an undertaking of the students, individually and collectively:

1. that they will not give or receive aid in examinations; that they will not give or receive un-permitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

2. that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

By writing my Magic Number below, I certify that I acknowledge and accept the Honor Code.

Magic Number---------------------------------

Computer Science Department
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Note: You may use a result you are asked to prove in subsequent questions even if you have not been able to prove it.

1. Norms and orthogonality [10pts]

1. [4pts] Let $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ be an orthonormal basis for R^n and $\mathbf{A}$ a $n \times n$ matrix. If $\mathbf{A}\mathbf{q}_1, \mathbf{A}\mathbf{q}_2, \ldots, \mathbf{A}\mathbf{q}_n$ is an orthonormal set as well, prove that $\mathbf{A}$ must be orthogonal.

2. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ nonsingular matrices satisfying $\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{B}\mathbf{x}\|_2$ for every $\mathbf{x} \in R^n$.

   (a) [3pts] Show that $\mathbf{A}$ and $\mathbf{B}$ have the same singular values.
   (*Hint:* Show that $\mathbf{A}^T\mathbf{A} = \mathbf{B}^T\mathbf{B}$)

   (b) [3pts] Show that $\mathbf{A} = \mathbf{Q}\mathbf{B}$ for an orthogonal matrix $\mathbf{Q}$.

2. Optimization and least squares [10 pts]

If $\mathbf{A}$ is an $m \times n$, $m > n$ matrix with full column rank and $\mathbf{b} \in R^m$, we know that the least squares solution to the overdetermined system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by the system of normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x}_0 = \mathbf{A}^T\mathbf{b}$ and corresponds to the vector $\mathbf{x}_0 \in R^n$ that minimizes $\phi(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$.

1. [4pts] Consider the modified functional

$$\hat{\phi}(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \mathbf{x}^T\mathbf{B}\mathbf{x} - 2\mathbf{c}^T\mathbf{x}$$

   where $\mathbf{B}$ is an $n \times n$ symmetric and positive definite matrix, and $\mathbf{c} \in R^n$. Show that we can find the value of $\mathbf{x}$ that minimizes $\hat{\phi}(\mathbf{x})$ by solving the following modification of the system of normal equations

$$\left(\mathbf{A}^T\mathbf{A} + \mathbf{B}\right)\mathbf{x}_0 = \mathbf{A}^T\mathbf{b} + \mathbf{c}$$

2. [1pt] Explain why we can find a symmetric, positive definite matrix $\mathbf{M}$ such that $\mathbf{B} = \mathbf{M}^2 = \mathbf{M}^T\mathbf{M}$.

3. [4pts] Show that the value of $\mathbf{x}$ that minimizes $\hat{\phi}(\mathbf{x})$ can alternatively be found by finding the least squares solution of the following modification of the original overdetermined system

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{M} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{b} \\ \mathbf{M}^{-1}c \end{pmatrix}$$

where $\mathbf{M}$ is the matrix defined in (3)

4. [1pts] Why would you prefer either of the approaches described in (1) or (3) to solve the given minimization problem?

3. Differential equations [10pts]

Consider the scalar ordinary differential equation $y' = \lambda y$, $\lambda \in R$ and the following methods for solving it

$$\begin{aligned} \text{Forward Euler} &: \quad y_{k+1} = y_k + hy'_k \\ \text{Backward Euler} &: \quad y_{k+1} = y_k + hy'_{k+1} \\ \text{Trapezoidal} &: \quad y_{k+1} = y_k + \frac{h}{2}(y'_k + y'_{k+1}) \end{aligned}$$

1. [4pts] Prove that taking one step of forward Euler to get from y_k to y_{k+1} followed by a step of backward Euler to get from y_{k+1} to y_{k+2} is equivalent to taking one trapezoidal step from y_k to y_{k+2} (note that the integration interval will be equal to $2h$ for a step from y_k to y_{k+2}).
2. [4pts] Prove that we get the same result if we use backward Euler for the first step and forward Euler for the second.
3. [2pts] Explain why both methods described in (1) and (2) are stable for any $\lambda < 0$