Comprehensive Examination in Logic

Stanford University

Department of Computer Science

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THE HONOR CODE:

(A) The honor code is an undertaking of the students individually and collectively :

(I) that they will not give or receive aid in examinations; they will not give or receive unauthorized aid in class work, in the preparation of reports, or in any other work that is to be used by the instructors as the basis of grading;

(II) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the honor code.

(B) The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. the faculty will avoid, as far as possible, academic procedures that create temptation to violate the Honour code.

(C) While the faculty alone have the right and the obligation to set academic requirements, the students and the faculty will work together to establish optimal conditions for honorable academic work.

By writing my "magic–number" below, I acknowledge and accept the honor code.

	CORRECT	INCORRECT	UNANSWERED	SCORE
SECTION A				
SECTION B				
TOTAL				

WRITE MAGIC NUMBER: ____________

Instructions

Please read these instructions and the *Notations* section carefully. **Do not read beyond this page until instructed to do so.**

- The exam is open book and open notes. But no laptops or electronic accessories are allowed.
- **All questions have penalties for wrong answers.** Read the instructions carefully before you start.
- Be sure to write your **magic number** on the previous page.

Notations

The notation is the one used by Enderton in *A Mathematical Introduction to Logic*, with the difference that the equality symbol is denoted by $==$ instead of $\approx$ and arguments to predicate and function symbols are enclosed in parentheses and separated by commas. Thus, for example, instead of Enderton's $fxyz$, $f(x, y, z)$ is used.
In some problems, the following symbols are used, whose definition is repeated here for completeness:

- $\mathrm{Cn}(\Gamma)$ is the set of logical consequences of an axiom set Γ;
- $\mathrm{Th}(\mathfrak{M})$ is the first-order theory of the structure $\mathfrak{M}$, i.e. the set of first-order sentences, of a given language, that are true in $\mathfrak{M}$.
- The composition of variable substitutions σ and τ is denoted by $\sigma \circ \tau$.

Do not turn this page until instructed to do so.

Section A

For each question in this section you need to choose one out of five choices provided. If you answer correctly you get **4 points.** Answering incorrectly will result in **3 points** being **deducted** from your score. Indicate your answer by writing your choice clearly in the box provided. **No points are deducted for leaving questions unanswered.**

1. Let $\mathcal{F}$ be a non-valid sentence. Which of the following statements are **possibly** true?

   I. $(\neg\ \mathcal{F})$ is valid.

   II. $(\neg\ \mathcal{F})$ is satisfiable.

   III. $\mathcal{F}$ is unsatisfiable.

   IV. $(\neg\ \mathcal{F})$ is unsatisfiable.

   (A) **II** only.

   (B) **I** and **II** only.

   (C) **I, II** and **III** only.

   (D) **II, III** and **IV** only.

   (E) all of them.

2. Consider deductive tableaux for propositional logic with resolution and the **"polarity strategy"**. Let ψ be a valid proposition. Which of the following statements are **necessarily** true?

   I. There may exist a proof of ψ which uses resolution but does not necessarily follow the polarity strategy.

   II. A tableau construction using resolution with the polarity strategy always terminates with a proof of ψ.

   III. There always exists a proof of ψ using resolution with the polarity strategy.

   IV. A tableau construction using resolution only, always terminates with a proof of ψ.

   (A) **I, II** and **III** only.

   (B) **II** and **IV** only.

   (C) **II** and **III** only.

   (D) **I** and **III** only.

   (E) all of them.

3. Which of the following sentences in first-order logic do **not** have a one-element model?

I. $(\forall x)(\exists y)R(x, y) \land (\forall x)[\neg R(x, x)]$.

II. $(\forall x)[R(x) \rightarrow (R(a) \lor R(b))]$.

III. $(\forall x)(\exists y)[R(x, x) \equiv \neg R(x, y)]$.

IV. $(\forall x)(\forall y)[R(y, x) \equiv \neg R(x, y)]$.

(A) **II** and **III** only.

(B) **II** and **IV** only.

(C) **III** and **IV** only.

(D) **I**, **III** and **IV** only.

(E) all of them.

4. Assume duality is added as a rule to the tableau. Which of the following sets of rules are complete for propositional logic tableau?

I. Duality, AA resolution without the polarity strategy.

II. Duality, GG resolution with the polarity strategy.

III. Duality, AA and GG resolutions without the polarity strategy.

IV. Duality with splitting rules.

(A) **II** only.

(B) **II** and **III** only.

(C) **II**, **III** and **IV** only.

(D) **I**, **III** and **IV** only.

(E) **I**, **II** and **III** only.

5. Given a theory $\mathcal{T}$ and a first-order formula ψ, if ψ is valid in $\mathcal{T}$, which of the following statements is **necessarily** true about $\mathcal{T}$ and ψ?

I. ψ is valid in first-order logic.

II. ψ is satisfiable in first-order logic.

III. $\mathcal{T}$ is consistent.

IV. For every model $\mathcal{M}$, if $\mathcal{M}$ satisfies $\mathcal{T}$ then $\mathcal{M}$ also satisfies ψ.

(A) **II** and **IV** only.

(B) **II** only.

(C) **II** and **III** only.

(D) **II**, **III** and **IV** only.

(E) **IV** only.

6. Let Σ be a **set** of first-order sentences. Which of the following statements about Σ are **necessarily** true?

I. For any ψ, if $\Sigma \models \psi$, then $\Sigma \not\models \neg\psi$.

II. If Σ is unsatisfiable then every subset of Σ is unsatisfiable.

III. If $\Sigma \models \psi$ and $\Sigma \models \neg\psi$ for some ψ then Σ does not have a model.

IV. If Σ is inconsistent then there is no ψ such that $\Sigma \models \psi$.

(A) **I** only.

(B) **III** only.

(C) **I** and **III** only.

(D) **II** and **IV** only.

(E) none of them.

7. Let $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ be first-order theories with $\mathcal{T}_1 \subseteq \mathcal{T}$ and $\mathcal{T}_2 \supseteq \mathcal{T}$. Which of the following statements are **necessarily** true?

I. If $\mathcal{T}$ is undecidable, then $\mathcal{T}_1$ is undecidable.

II. If $\mathcal{T}$ is undecidable, then $\mathcal{T}_2$ is undecidable.

III. If $\mathcal{T}$ is decidable, then $\mathcal{T}_1$ is decidable.

IV. If $\mathcal{T}$ is decidable, then $\mathcal{T}_2$ is decidable.

(A) **II** only.

(B) **III** only.

(C) **II** and **III** only.

(D) **I** and **IV** only.

(E) none of them.

8. Let $\mathcal{F}$ be a valid sentence in predicate logic. Which of the following statements are **necessarily** true?

I. $\mathcal{F}$ is valid in the theory of natural numbers with 0,1, addition and multiplication.

II. $\mathcal{F}$ is valid in the theory of natural numbers **with** 0,1, addition and **without** multiplication.

III. $\mathcal{F}$ is true in every model of the theory of natural numbers **with** 0, 1, addition and **without** multiplication.

IV. There are **first–order** theories $\mathcal{T}$ such that $\mathcal{T} \not\models \mathcal{F}$.

(A) **III** only.

(B) **II** and **III** only.

(C) **I, II** and **III** only.

(D) **II, III** and **IV** only.

(E) all of them.

9. Which of the following variable substitutions are the most general unifiers (mgu's) of x and y?

I. $\{x \leftarrow y\}$.

II. $\{x \leftarrow y, y \leftarrow x\}$.

III. $\{x \leftarrow z, y \leftarrow z\}$.

IV. $\{x \leftarrow z, y \leftarrow z, z \leftarrow y\}$.

(A) **I** only.

(B) **I** and **IV** only.

(C) **I**, **III** and **IV** only.

(D) **I**, **II** and **III** only.

(E) all of them.

10. Let $\mathcal{F}$ and $\mathcal{G}$ be two expressions. Which of the following statements about unifiers of $\mathcal{F}$ and $\mathcal{G}$ are **necessarily** true? (Recall that $\circ$ denotes the composition operation of variable substitutions.)

I. If θ is a mgu, and θ' is a permutation, then $\theta \circ \theta'$ is a mgu.

II. If both θ and θ' are mgu's, then there exists a permutation θ'' such that $\theta \circ \theta'' = \theta'$.

III. If both θ and θ' are mgu's, then $\theta \circ \theta'$ is also a mgu.

IV. If neither θ nor θ' is a mgu, then $\theta \circ \theta'$ can not be a mgu.

(A) **I**, **II** and **III** only.

(B) **I** only.

(C) **II** only.

(D) **I** and **II** only.

(E) **I** and **IV** only.

11. Let $\mathcal{F}$ and $\mathcal{G}$ be sentences in first-order logic. Given that $\mathcal{F}$ is valid if and only if $\mathcal{G}$ is **not** valid, which of the following statements are **necessarily** true?

I. $(\mathcal{F} \not\equiv \mathcal{G})$ is valid.

II. $(\mathcal{F} \not\equiv \mathcal{G})$ is satisfiable.

III. $(\mathcal{F} \equiv \mathcal{G})$ is unsatisfiable.

IV. $(\mathcal{F} \equiv \mathcal{G})$ is not valid.

(A) **II** and **IV** only.

(B) **I** and **III** only.

(C) **II** only.

(D) **IV** only.

(E) none of them.

12. Given a sentence $\mathcal{F}$, we obtain $\mathcal{G}$ by removing all its quantifiers through **validity preserving** skolemization. Which of the following statements are **necessarily** true about $\mathcal{F}$ and $\mathcal{G}$?

I. $\neg\mathcal{G}$ is satisfiable if and only if $\neg\mathcal{F}$ is satisfiable.

II. $\mathcal{G}$ may not exist for any given $\mathcal{F}$.

III. It is always possible to skolemize in such a way that $\mathcal{G}$ and $\mathcal{F}$ are equivalent.

IV. If $\mathcal{F}$ and $\mathcal{G}$ are not equivalent, then $\mathcal{F}$ is not valid.

(A) **IV** only.

(B) **I** and **IV** only.

(C) **III** only.

(D) **II** only.

(E) **II** and **III** only.

13. Which of the following statements is **necessarily** true about a first-order theory $\mathcal{T}$?

I. If $\mathcal{T}$ is finitely axiomatizable, then $\mathcal{T}$ is decidable.

II. If $\mathcal{T}$ is not finitely axiomatizable, then $\mathcal{T}$ is undecidable.

III. If $\mathcal{T}$ is not finitely axiomatizable, then $\mathcal{T}$ does not have a finite model.

IV. If $\mathcal{T}$ is finitely axiomatizable, then $\mathcal{T}$ has a finite model.

(A) **I** and **III** only.

(B) **II** and **IV** only.

(C) **IV** only.

(D) **III** only.

(E) none of them.

14. Consider the sentence

$$(\forall x)(\exists y)[(\exists z)(\forall w)\varphi(x,y,z,w) \rightarrow (\forall z)(\exists w)\psi(x,y,z,w)].$$

Which of the following can appear as the results of one or more steps of **validity-preserving** skolemization?

I. $(\exists y)[(\exists z)(\forall w)\varphi(a,y,z,w) \rightarrow (\forall z)(\exists w)\psi(a,y,z,w)]$.

II. $(\exists y)[(\forall w)\varphi(a,y,f(y),w) \rightarrow (\forall z)(\exists w)\psi(a,y,z,w)]$.

III. $(\exists y)[(\forall w)\varphi(a,y,f(y),w) \rightarrow (\exists w)\psi(a,y,g(y),w)]$.

IV. $(\forall x)(\exists y)[(\forall w)\varphi(x,y,f(y),w) \rightarrow (\exists w)\psi(x,y,g(y),w)]$.

(A) **I** only.

(B) **III** only.

(C) **I** and **III** only.

(D) **I**, **II** and **III** only.

(E) all of them.

15. Consider the following terms:

$$\begin{aligned} t_1 &: \; f(x, g(x)) \\ t_2 &: \; f(y, g(y)) \\ t_3 &: \; f(g(z), g(y)) \end{aligned}$$

Which of the following statements are **necessarily** true?

I. t_1 and t_2 are unifiable.

II. t_2 and t_3 are unifiable.

III. t_1 and t_3 are unifiable.

IV. The tuple (t_1, t_2, t_3) is unifiable.

(A) **I** only.

(B) **I** and **II** only.

(C) **I** and **III** only.

(D) **I**, **II** and **III** only.

(E) all of them.

16. Let $\mathcal{F}, \mathcal{G}$ be arbitrary predicate logic formulas, and $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ be defined as follows:

$$\begin{aligned} \mathcal{S}_1 &: \; (\forall *)(\mathcal{F} \rightarrow \mathcal{G}) \\ \mathcal{S}_2 &: \; (\forall *)\mathcal{F} \rightarrow (\forall *)\mathcal{G} \\ \mathcal{S}_3 &: \; (\exists *)(\mathcal{F} \rightarrow (\forall *)\mathcal{G}) \end{aligned}$$

Which of the following statements are **necessarily** true?

I. $\mathcal{S}_1$ implies $\mathcal{S}_2$.

II. If $\mathcal{S}_1$ is valid, then $\mathcal{S}_2$ is also valid.

III. If $\mathcal{S}_3$ is valid, then $\mathcal{S}_1$ is also valid.

IV. If $\mathcal{S}_1$ is valid, then $\mathcal{S}_3$ is also valid.

(A) **I** and **II** only.

(B) **I** and **III** only.

(C) **II** only.

(D) **I**, **II** and **IV** only.

(E) all of them.

17. Which of the following can appear as the result of an application of AG or GA resolution rule between assertion

$$\mathcal{A}: \qquad \big(p(x,y) \vee q(a,a)\big) \rightarrow q(y,b)$$

and goal

$$\mathcal{G}: \qquad q(a,z) \vee r(z)?$$

I. $\neg(p(x,y) \rightarrow q(y,b))$.

II. $p(x,a) \vee q(a,a)$.

III. $r(b)$.

IV. $r(a) \wedge (\neg q(y,b))$.

(A) **I**, **II** and **IV** only.

(B) **I**, **II** and **III** only.

(C) **I**, **III** and **IV** only.

(D) **II**, **III** and **IV** only.

(E) all of them.

Section B

For each question in this section you need to write "yes" if you think the statement holds or "no" if you think it does not. You receive **2** points if you answer correctly. However **2** points will be **deducted** for wrong answers. There is no penalty for leaving questions unanswered.

18. $(\exists x)\big(\varphi(x) \to (\forall y)\varphi(y)\big)$ is a valid first-order sentence.

19. $(\exists x)(\forall y)(\exists z)(x + y = z)$ is a valid first-order sentence.

20. If a sentence φ is only satisfiable in infinite models, then $\neg\,\varphi$ is satisfiable.

21. Any first-order theory which is recursive is axiomatizable.

22. If $\mathcal{A} = \langle A, < \rangle$ is a well-founded structure, then there is no infinite chain of the form

$$\ldots < a_{-i} < \ldots < a_{-1} < a_0 < a_1 < \ldots < a_i < \ldots$$

23. Let $\mathcal{F}$ and $\mathcal{G}$ be two first-order unifiable expressions. Then there exist infinitely many mgu's of $\mathcal{F}$ and $\mathcal{G}$.

24. There are infinitely many idempotent variable permutations.

25. A tuple of three expressions is unifiable if and only if any two of those expressions are unifiable.

26. It is not possible to write a first-order sentence $\mathcal{F}$ such that $\mathcal{F}$ is true and only true in all finite models.

27. Recall that $\mathrm{Th}(\mathfrak{M})$ denotes the first-order theory of the structure $\mathfrak{M}$. Let $\mathfrak{N} = \langle N, 0, 1, +, * \rangle$ denote the structure of natural numbers with addition and multiplication. Then $\mathrm{Th}(\mathfrak{N})$ is a complete theory.

28. If $\mathcal{T}$ is a complete first-order theory, then for every cardinality κ, $\mathcal{T}$ has, up to isomorphism, exactly one model.

29. $\{x \leftarrow f(x), y \leftarrow x\}$ is a mgu of x and $f(y)$.

30. If θ is a mgu of $\mathcal{F}$ and $\mathcal{G}$, and τ is **not** a variable permutation, then $\theta \circ \tau$ can not be a mgu of $\mathcal{F}$ and $\mathcal{G}$.