Stanford University
Computer Science Department

Fall 2003 Comprehensive Exam in Logic

1. **Open Book & Notes / No Laptops. Write only in the space provided on the question paper.**
2. **The exam is timed for 60 minutes.**
3. **Write your Magic Number on this sheet.**

The following is a statement of the Stanford University Honor Code:

A. The Honor Code is an undertaking of the students, individually and collectively:

1. that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

2. that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

By writing my Magic Number below, I certify that I acknowledge and accept the Honor Code.

Magic Number ---------------------------------

Comprehensive Examination in Logic

Stanford University

November 2003

Instruction

- You have **60 minutes** to complete the exam.
- The exam consists of **20 questions** for a **total of 60 points**. You get **3 points** for each correct answer and **-1 point** for each incorrect one.
- The exam is **open book**, but no laptops or electronics accessories are allowed.

Do not turn this page until instructed to do so.

Proposition Logic (9 points)

Question 1. The sentence $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$ is:

(A) Valid and satisfiable.

(B) Invalid but satisfiable.

(C) Valid but unsatisfiable.

(D) Invalid and unsatisfiable.

Question 2. Let P stand for *"I will serve you tea"* and let Q stand for *"you order coffee"*. The sentence *"I will serve you tea if you do not order coffee"* is best represented by which of the following?

(A) $\neg P \rightarrow Q$.

(B) $P \rightarrow \neg Q$.

(C) $P \rightarrow Q$.

(D) $\neg P \rightarrow \neg Q$.

Question 3. Which of the following forms a complete set of propositional connectives?

I. $\wedge, \neg$

II. $\wedge, \vee$

III. $\rightarrow$, *false*

(A) I only.

(B) II only.

(C) I and III only.

(D) I, II, and III.

Predicate Logic (9 points)

Question 4. Consider the sentences:

I. $(\exists y)(\forall x)P(x, y) \rightarrow (\forall x)(\exists y)P(x, y)$

II. $(\forall x)(\exists y)P(x, y) \rightarrow (\exists y)(\forall x)P(x, y)$

III. $(\forall x)Q(x) \rightarrow (\exists x)Q(x)$

Which of them is valid?

(A) None.

(B) I and III only.

(C) II and III only.

(D) III only.

Question 5. Let $\mathcal{F}$ and $\mathcal{G}$ be arbitrary formulae, and define S_1 and S_2 as follows:

$$S_1 : \quad (\forall x)\mathcal{F} \rightarrow (\forall x)\mathcal{G}$$
$$S_2 : \quad (\forall x)(\mathcal{F} \rightarrow \mathcal{G})$$

Which of the following are true?

I. S_1 and S_2 are equivalent.

II. Assuming that S_1 and S_2 are closed, if S_1 is valid then S_2 is valid.

III. Assuming that S_1 and S_2 are closed, if S_2 is valid then S_1 is valid.

(A) I only.

(B) II only.

(C) III only.

(D) I and III only.

Question 6. Let $\mathcal{F}$ and $\mathcal{G}$ be arbitrary formulae. If $(\exists x)(\mathcal{F} \wedge \neg\mathcal{G})$ is unsatisfiable then:

(A) $(\exists x)(\neg\mathcal{F} \rightarrow \mathcal{G})$ is valid.

(B) $(\forall x)(\mathcal{F} \wedge \mathcal{G})$ is valid.

(C) $(\forall x)(\mathcal{F} \rightarrow \mathcal{G})$ is valid.

(D) $(\forall x)(\mathcal{G} \rightarrow \mathcal{F})$ is valid.

Unification (6 points)

Question 7. Which of the following is a most general unifier of $h(x,y)$ and $h(g(y), f(x))$?

(A) $\{x \leftarrow g(y)\}$

(B) $\{y \leftarrow f(x)\}$

(C) $\{x \leftarrow g(y), y \leftarrow f(x)\}$

(D) the terms are not unifiable.

Question 8. Which of the following are unifiable?

$$t_1 : h(f(x), x)$$
$$t_2 : h(y, z)$$
$$t_3 : h(z, g(y))$$

(A) (t_1, t_2) only.

(B) (t_1, t_2) and (t_1, t_3) only.

(C) (t_1, t_2) and (t_2, t_3) only.

(D) (t_2, t_3) and (t_1, t_3) only.

Skolemization (6 points)

Question 9. Consider the sentence

$$(\forall x)(\exists y)[(\exists z)(\forall w)P(x,y,z,w) \rightarrow (\exists w)Q(w)]\,.$$

Which of the following skolemizations preserves *validity*?

(A) $(\forall x)[(\exists z)P(x, f(x), z, g(x,z)) \rightarrow Q(h(x))]$

(B) $(\exists y)[(\forall w)P(a, y, f(y), w) \rightarrow (\exists w)Q(w)]$

(C) $(\forall x)[(\exists z)P(x, f(x), z, g(x,z)) \rightarrow Q(a)]$

(D) $(\forall x)[(\forall w)P(x, f(x), g(x,w), w) \rightarrow Q(h(x))]$

Question 10. Consider the sentence

$$(\forall x)(\exists y)[(\exists z)(\forall w)P(x,y,z,w) \rightarrow (\exists w)Q(w)]\,.$$

Which of the following skolemizations preserves *satisfiability*?

(A) $(\forall x)[(\exists z)P(x, f(x), z, g(x,z)) \rightarrow Q(h(x))]$

(B) $(\exists y)[(\forall w)P(a, y, f(y), w) \rightarrow (\exists w)Q(w)]$

(C) $(\forall x)[(\exists z)P(x, f(x), z, g(x,z)) \rightarrow Q(a)]$

(D) $(\forall x)[(\forall w)P(x, f(x), g(x,w), w) \rightarrow Q(h(x))]$

Deductive tableaux (6 points)

Question 11. Consider the following deductive tableau where x, y, and z are variables, and a is a constant.

	assertions	goals
A1	$P(x,x) \vee Q(y)$	
G2		$R(y,z) \wedge P(a,z)$

What is the result of an AG-resolution of A1 and G2?

(A)

G3		$\neg Q(y) \wedge R(y,z)$

(B)

G3		$\neg Q(y) \vee R(w,a)$

(C)

G3		$Q(y) \wedge R(w,z)$

(D)

G3		$\neg Q(y) \wedge R(w,a)$

Question 12. Consider the connective $\downarrow$ (*nor*) with the following semantics: the truth value of $\mathcal{F} \downarrow \mathcal{G}$ is true if both $\mathcal{F}$ and $\mathcal{G}$ are false, and false otherwise. Which of the following nor-splitting deduction rules preserves equivalence of a deductive tableau?

I.

assertions	goals
$A_1 \downarrow A_2$	
	A_1
	A_2

II.

assertions	goals
$A_1 \downarrow A_2$	
A_1	
A_2	

III.

assertions	goals
	$A_1 \downarrow A_2$
A_1	
A_2	

(A) None.

(B) I only.

(C) II only.

(D) I and III only.

Polarity (6 points)

Question 13. Let X be a formula whose occurrences in $\mathcal{F}[X]$ have all negative polarity. Assume that the formula $X \to Y$ is true under interpretation I. Then:

(A) $\mathcal{F}[X]$ is true under I.

(B) $\mathcal{F}[Y]$ is true under I.

(C) $\mathcal{F}[X] \to \mathcal{F}[Y]$ is true under I.

(D) $\mathcal{F}[Y] \to \mathcal{F}[X]$ is true under I.

Question 14. Which of the following sentences are instances of the polarity proposition?

I. $(P \to Q) \to ((R \to P) \to (R \to Q))$

II. $(Q \to P) \to ((R \to P) \to (R \to Q))$

III. $(P \to Q) \to (Q \to P)$

(A) None.

(B) I only.

(C) II only.

(D) III only.

First-order theories (9 points)

Question 15. Let T be a first-order theory such that

- all finite interpretations with an even number of members are models of T;
- all finite interpretations with an odd number of members are not models of T.

Which of the following hold?

I. T does not exists

II. T has an enumerable model

III. There exists a theory S such that $T \cup S \models (\forall x, y)(x = y)$

(A) None.

(B) I only.

(C) II only.

(D) II and III only.

Question 16. Let T be a finitely axiomatizable theory. Which of the following necessarily holds?

I. T has a finite model

II. T is decidable

III. If T is complete, then T is decidable

(A) None.

(B) I and II only.

(C) I and III only.

(D) III only.

Question 17. Which of the following theories is undecidable?

(A) The theory of natural numbers under zero and successor.

(B) The theory of natural numbers under zero, successor, and addition.

(C) The theory of natural numbers under zero, successor, addition, and multiplication.

(D) None of the above.

Well-founded relations (9 points)

Question 18. Let $<$ be a well-founded relation over A. Which of the following necessarily holds?

(A) $<$ is irreflexive.

(B) $<$ is transitive.

(C) A is finite.

(D) A is infinite.

Question 19. Which of the following is true?

I. There exists a well-founded relation which is symmetric

II. There exists a well-founded relation which is transitive

(A) None.

(B) I only.

(C) II only.

(D) I and II.

Question 20. Which of the following relations $\prec$ over the set of natural numbers is well-founded?

I. $x \prec y$ iff $x = y + 1$

II. $x \prec y$ iff $x \leq y$

III. $x \prec y$ iff there exists a z such that $yz = x$.

(A) I only.

(B) I and II only.

(C) I and III only.

(D) None of the above.