Comprehensive Examination in Logic

Stanford University
Department of Computer Science

November 2002

Instructions

Please read these instructions and the *Notation* section carefully. Do not read beyond this page until instructed to do so.

You should mark your answers only in the **answer sheet** that is provided with this part of the Comprehensive Examination. Be sure to write your **magic number** on the answer sheet.

This exam is **open book** and is composed of **44 questions** on **8 pages**, plus one answer sheet. For each question, write either **YES** or **NO** in the corresponding box of the answer sheet, or leave it blank. You will receive +1 point for each correct answer, −1 point for each incorrect answer, and **0** points for a blank (or crossed out) answer. You have **60 minutes** to complete the exam.

Notation

The notation is the one used by Enderton in *A Mathematical Introduction to Logic*, with the difference that the equality symbol is denoted by $=$ instead of $\approx$ and arguments to predicate and function symbols are enclosed in parentheses and separated by commas. Thus, for example, instead of Enderton's $fxyz$, $f(x,y,z)$ is used.

In particular, recall that "theory" means "a set of sentences closed by logical consequence". The "theory of a model $\mathfrak{M}$" is the set of all sentences true in $\mathfrak{M}$.

Do not turn this page until instructed to do so.

An AI system for error diagnosis works by keeping a knowledge base, about the functioning of the artifact it diagnoses, as a set of sentences of first-order logic. When a set of observable behaviors are given as inputs, again as sentences of first-order logic, we can ask whether a certain component is necessarily broken. The system uses a first-order reasoning algorithm to look for a proof of this fact. Replacing a working component of the artifact is inadmissible. for cost reasons, but we can admit some undetected broken components. Then:

1. The reasoning algorithm should necessarily be sound.

2. The reasoning algorithm should necessarily be complete.

Answer "yes" or "no".

3. Is $((p \rightarrow q) \rightarrow q) \rightarrow q$ a tautology of propositional logic?

4. Suppose you are given a machine that you can feed any propositional formula and tells you, in constant time, whether that formula is a tautology. Could you then build a machine that decides validity of first-order sentences?

Assume $\alpha \rightarrow \beta \vee \gamma$ is valid. Then necessarily:

5. $\alpha \wedge \neg\beta \rightarrow \gamma$ is valid?

6. $\alpha \vee \beta \vee \gamma$ is satisfiable?

Which of the following are valid formulas of first-order logic?

7. $\forall x\ p(x) \vee \forall x\ q(x) \rightarrow \forall x\ (p(x) \vee q(x))$?

8. $\forall x\ (p(x) \vee q(x)) \rightarrow \forall x\ p(x) \vee \forall x\ q(x)$?

9. $\forall y\ (\ \exists x\ (p(y) \wedge q(x)) \leftrightarrow p(y) \wedge \exists x\ q(x)\ )$?

Consider the first-order language with equality and one binary predicate symbol R. A model of this language can be seen as a graph.

10. Can the property of a graph G *"There is a path of length at most 2 between any given two vertices of G."* be expressed with a first-order sentence over this language?

11. Does the following sentence express the property *"G is symmetric and transitive"*?

$$\forall x\,\forall y\,\forall z\;[[R(x,y) \land \neg R(y,x) \rightarrow \neg R(x,y)] \land [R(z,x) \land R(z,y) \rightarrow R(y,x)]]$$

12. Does the following sentence express the property *"G is acyclic"*?

$$\forall x\,\forall y\,\forall z\;[\neg R(x,x) \land [R(z,x) \land R(z,y) \rightarrow R(y,x)]]$$

Consider validity-preserving skolemization.

13. Does $\exists x\;(p(x,x) \rightarrow \forall y\,\exists z\;(q(x,y,z) \lor p(z,x)))$ correctly skolemize to $p(x,x) \rightarrow q(x,f(x),z) \lor p(z,x)$?

14. Does $\forall x\;\exists y\;(y > x \land \neg\exists u\;\exists v\;(\neg u = y \land \neg u = 1 \land u \cdot v = y))$ skolemize to $y > a \land \neg(\neg f(y) = y \land \neg f(y) = 1 \land u \cdot v = y)$? ($\mathbf{1}, a, b, c$ are constants, u, v, x, y, z are variables, etc.: no tricks).

Consider the following terms t_1, t_2, and t_3:

t_1: $f(h(r,s,t), h(w,x,y), t, w)$
t_2: $f(h(g(u,v),r,s), h(x,y,w), g(v,a), v)$
t_3: $f(h(g(u,v),s,w), h(x,y,w), t, t)$

where $r, s, t, u, v, w, x, y, z$ are variables and a is a constant symbol. Which of the following sets are unifiable?

15. $\{t_1, t_2\}$

16. $\{t_1, t_3\}$

17. $\{t_2, t_3\}$

18. $\{t_1, t_2, t_3\}$

Consider the following deductive tableau, where x, y, and z are variables, and a is a constant:

	A	G
1		$p(a,x) \vee p(x,a)$
2	$p(z,y) \rightarrow p(y,y)$	

Which of the following rows are results of an application of resolution according to the polarity strategy?

19.

		$\perp$

20.

		$p(a,a)$

21.

		$p(z,a)$

Let C be the first-order theory of complex numbers with addition, multiplication.

22. Does C have a finite model?

23. Does C have a countably infinite model?

24. Does C have an uncountably infinite model?

Consider the first-order theory, N, of the standard model of natural numbers with addition and multiplication.

25. Is N complete?

26. Is N decidable?

Let φ be an arbitrary sentence of first-order logic, and T an arbitrary axiomatizable theory over the same language as φ. Assume you have a resolution theorem prover that follows a resolution strategy S: you give the prover the axioms of T (including the Reflexivity of Equality axiom) and try to prove φ; the prover starts producing resolvents. Five weeks later it reports that no new resolvent can be produced and no proof has been found. Which of the following statements are correct?

27. A faster prover, or running the prover for a longer time, might find a proof of φ.

28. Either S is incomplete or T is incomplete.

29. If S is complete and T is complete, then T contains the sentence $\neg\varphi$.

Is there a first-order theory (with equality) that:

30. has exactly one model up to isomorphism?

31. has exactly one infinite model up to isomorphism?

32. has exactly two infinite models up to isomorphism?

Does the following hold?

33. For a set of first-order sentences, A, and a sentence φ, $A \models \varphi$ if and only if there is a finite subset B of A such that $B \models \varphi$.

34. A theory T is satisfiable if and only if it has a finite satisfiable subset.

35. Every theory is a subset of some consistent theory.

36. Let φ be a sentence that has no finite models. Then $\neg\varphi$ has a countably infinite model.

Does the following hold?

37. Let T be an incomplete theory, over a countable language, with infinite models and no finite models. Then there are at least two non-isomorphic models of T for every infinite cardinality.

38. There is a first-order sentence φ_{inf} such that, for every model $\mathfrak{M}$, $\mathfrak{M} \models \varphi_{\text{inf}}$ if and only if $|\mathfrak{M}|$ is infinite.

39. If $\sqsubset$ is well-founded over the set A, then every nonempty subset of $A \times A$ has a minimal element according to the relation $\prec$ defined by $(a, b) \prec (a', b') \Leftrightarrow a \sqsubset a'$ or $b \sqsubset b'$.

Answer "yes" or "no". In the context of deductive tableaux:

40. Is there a first-order theory for which resolution alone, without skolemization, is complete for computing validity?

41. Is there a first-order theory for which skolemization alone, without resolution, is complete for computing validity?

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42. Let A be a set of first-order sentences and $\varphi[(\exists x\ \psi)^{-}] \in A$ (the minus sign denotes polarity). Let $A' = (A \setminus \{\varphi[(\exists x\ \psi)^{-}]\}) \cup \{\varphi[(\forall x\ \psi)^{-}]\}$. Is it true that if A is inconsistent, then A' is inconsistent?

Suppose you are visiting a forest in which every inhabitant is either a knight or a knave. Knights always speak the truth and knaves always lie.

43. You witness the following conversation among three inhabitants A, B, and C:

A: At least one of the three of us is a knave.
B: C is a knight.

Do you now know for sure who among A, B, and C are the knights?

44. Inspector Craig of Scotland Yard was called to the Forest of Knights and Knaves to help find a criminal named Arthur York. What made the process difficult was that it was not known whether Arthur York was a knight or a knave.

One suspect was arrested and brought to trial. Inspector Craig was the presiding Judge. Here is a transcript of the trial:

> CRAIG: What do you know about Arthur York?
> DEFENDANT: Arthur York once claimed that I was a knave.
> CRAIG: Are you by any chance Arthur York?
> DEFENDANT: Yes.

Is the defendant Arthur York?

ANSWER SHEET

Comprehensive Examination in LOGIC

November 200[illegible]

MAGIC NUMBER: __________

1		2		3		4		5	
6		7		8		9		10	
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41		42		43		44			

THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code. (Signed) ____________________