Comprehensive Examination in Logic

Stanford University

Department of Computer Science

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Solutions

Instructions

Please read these instructions and the *Notation* section carefully. Do not read beyond this page until instructed to do so.

You should mark your answers only in the **answer sheet** that is provided with this part of the Comprehensive Examination. Be sure to write your **magic number** on the answer sheet.

This exam is **open book** and is composed of **44 questions** on **8 pages**, plus one answer sheet. For each question, write either **YES** or **NO** in the corresponding box of the answer sheet, or leave it blank. You will receive **+1** point for each correct answer, **−1** point for each incorrect answer, and **0** points for a blank (or crossed out) answer. You have **60 minutes** to complete the exam.

Notation

The notation is the one used by Enderton in *A Mathematical Introduction to Logic*, with the difference that the equality symbol is denoted by $=$ instead of $\approx$ and arguments to predicate and function symbols are enclosed in parentheses and separated by commas. Thus, for example, instead of Enderton's $fxyz$, $f(x, y, z)$ is used.

In particular, recall that "theory" means "a set of sentences closed by logical consequence". The "theory of a model $\mathfrak{M}$" is the set of all sentences true in $\mathfrak{M}$.

Do not turn this page until instructed to do so.

An AI system for error diagnosis works by keeping a knowledge base, about the functioning of the artifact it diagnoses, as a set of sentences of first-order logic. When a set of observable behaviors are given as inputs, again as sentences of first-order logic, we can ask whether a certain component is necessarily broken. The system uses a first-order reasoning algorithm to look for a proof of this fact. Replacing a working component of the artifact is inadmissible, for cost reasons, but we can admit some undetected broken components. Then:

1. The reasoning algorithm should necessarily be sound.

Answer. YES. Soundness means every sentence that is proved follows from the assumptions, which is what we need to prevent detecting unexisting errors.

2. The reasoning algorithm should necessarily be complete.

Answer. NO. Completeness means every sentence that follows from the assumptions is provable, but we can admit some undetected broken components, thus completeness is not required.

Answer "yes" or "no".

3. Is $((p \rightarrow q) \rightarrow q) \rightarrow q$ a tautology of propositional logic?

Answer. NO. The falsifying valuation, obtainable by the falsification method, makes p true and q false.

4. Suppose you are given a machine that you can feed any propositional formula and tells you, in constant time, whether that formula is a tautology. Could you then build a machine that decides validity of first-order sentences?

Answer. NO. This would imply decidability of validity for first-order logic.

Assume $\alpha \rightarrow \beta \vee \gamma$ is valid. Then necessarily:

5. $\alpha \wedge \neg\beta \rightarrow \gamma$ is valid?

Answer. YES. Consider an arbitrary interpretation. If $\alpha \wedge \neg\beta$ holds in that interpretation, then since $\alpha \rightarrow \beta \vee \gamma$ holds by assumption, $\beta \vee \gamma$ holds, and therefore, since β does not hold, γ does. Thus $\alpha \wedge \neg\beta \rightarrow \gamma$ holds in every interpretation.

6. $\alpha \vee \beta \vee \gamma$ is satisfiable?

Answer. NO. Consider for example $\alpha \equiv \beta \equiv \gamma \equiv p \wedge \neg p$.

Which of the following are valid formulas of first-order logic?

7. $\forall x\ p(x) \vee \forall x\ q(x) \rightarrow \forall x\ (p(x) \vee q(x))$?

Answer. YES.

8. $\forall x\ (p(x) \vee q(x)) \rightarrow \forall x\ p(x) \vee \forall x\ q(x)$?

Answer. NO.

9. $\forall y\ (\ \exists x\ (p(y) \wedge q(x)) \leftrightarrow p(y) \wedge \exists x\ q(x)\ )$?

Answer. YES.

Consider the first-order language with equality and one binary predicate symbol R. A model of this language can be seen as a graph.

10. Can the property of a graph G *"There is a path of length at most 2 between any given two vertices of G."* be expressed with a first-order sentence over this language?

Answer. YES. For example,
$\forall x\, \forall y\, (x = y \vee R(x,y) \vee \exists z\, (R(x,z) \wedge R(z,y)))$.

11. Does the following sentence express the property *"G is symmetric and transitive"*?

$$\begin{array}{l}\forall x\, \forall y\, \forall z\, [[R(x,y) \wedge \neg R(y,x) \rightarrow \neg R(x,y)] \\ \quad \wedge \\ \quad [R(z,x) \wedge R(z,y) \rightarrow R(y,x)]]\end{array}$$

Answer. YES. The first conjunct is equivalent to $R(x,y) \rightarrow R(y,x)$, i.e., symmetry, in view of the propositional tautology $(p \rightarrow q) \leftrightarrow (p \wedge \neg q \rightarrow\ p)$. The second conjunct is equivalent to transitivity once symmetry is assumed, because $R(z,y)$ is then equivalent to $R(y.z)$.

12. Does the following sentence express the property *"G is acyclic"*?

$$\forall x\, \forall y\, \forall z\, [\neg R(x,x) \wedge [R(z,x) \wedge R(z,y) \rightarrow R(y,x)]]$$

Answer. NO. In fact, acyclicity cannot be expressed in first-order logic.

Consider validity-preserving skolemization.

13. Does $\exists x\, (p(x,x) \rightarrow \forall y\, \exists z\, (q(x,y,z) \vee p(z,x)))$ correctly skolemize to $p(x,x) \rightarrow q(x, f(x), z) \vee p(z,x)$?

Answer. YES.

14. Does $\forall x\, \exists y\, (y > x \wedge \neg \exists u\, \exists v\, (\neg u = y \wedge \neg u = 1 \wedge u \cdot v = y))$ skolemize to $y > a \wedge \neg(\neg f(y) = y \wedge \neg f(y) = 1 \wedge u \cdot v = y)$? ($1, a, b, c$ are constants, u, v, x, y, z are variables, etc.: no tricks).

Answer. NO. The last u and v should be $f(y)$ and $g(y)$, respectively.

Consider the following terms t_1, t_2, and t_3:

$$t_1:\ f(h(r,s,t), h(w,x,y), t, w)$$
$$t_2:\ f(h(g(u,v),r,s), h(x,y,w), g(v,a), v)$$
$$t_3:\ f(h(g(u,v),s,w), h(x,y,w), t, t)$$

where $r, s, t, u, v, w, x, y, z$ are variables and a is a constant symbol. Which of the following sets are unifiable?

15. $\{t_1, t_2\}$

Answer. YES.

16. $\{t_1, t_3\}$

Answer. YES.

17. $\{t_2, t_3\}$

Answer. NO.

18. $\{t_1, t_2, t_3\}$

Answer. NO.

Consider the following deductive tableau, where x, y, and z are variables, and a is a constant:

	A	G
1		$p(a,x) \vee p(x,a)$
2	$p(z,y) \rightarrow p(y,y)$	

Which of the following rows are results of an application of resolution according to the polarity strategy?

19.

		$\bot$

Answer. NO. This resolvent can be obtained only by violating the polarity strategy.

20.

		$p(a,a)$

Answer. NO. Obtaining this resolvent would involve using unifier that is not most general.

21.

		$p(z,a)$

Answer. YES. Resolve the two (positive) p-atoms in Goal 1 and the (negative) p-atom in the consequent of Assertion 2, with most general unifier $\{x \leftarrow a, y \leftarrow a\}$. In fact, this is the only resolvent according to the polarity strategy.

Let C be the first-order theory of complex numbers with addition, multiplication.

22. Does C have a finite model?

Answer. NO. The theory contains a formula saying there are n different elements in the domain, for every n, thus every domain must be infinite.

23. Does C have a countably infinite model?

Answer. YES. By the Löwenheim-Skolem-Tarski Theorem, since the language is finite and there is an uncountably infinite model (next question).

24. Does C have an uncountably infinite model?

Answer. YES. The complex numbers are one.

Consider the first-order theory, N, of the standard model of natural numbers with addition and multiplication.

25. Is N complete?

Answer. YES. The theory of any single structure is complete.

26. Is N decidable?

Answer. NO. If it were, it would be an axiomatizable extension of Peano Arithmetic, thus violating Gödel's Incompleteness Theorem.

Let φ be an arbitrary sentence of first-order logic, and T an arbitrary axiomatizable theory over the same language as φ. Assume you have a resolution theorem prover that follows a resolution strategy S; you give the prover the axioms of T (including the Reflexivity of Equality axiom) and try to prove φ; the prover starts producing resolvents. Five weeks later it reports that no new resolvent can be produced and no proof has been found. Which of the following statements are correct?

27. A faster prover, or running the prover for a longer time, might find a proof of φ.

Answer. NO.

28. Either S is incomplete or T is incomplete.

Answer. NO. If S is any complete strategy and T is any complete consitent theory, a φ such that $T \models \neg\varphi$ will do.

29. If S is complete and T is complete, then T contains the sentence $\neg\varphi$.

Answer. YES. Since S is complete, φ is not in the theory T (otherwise, the prover would eventually produce a proof, but none has been found and no new resolvent can be produced). Therefore, since T is complete, $\neg\varphi$ must be in the theory T.

Is there a first-order theory (with equality) that:

30. has exactly one model up to isomorphism?

Answer. YES. For example, the theory with only the axiom $\forall x\, \forall y\; x = y$.

31. has exactly one infinite model up to isomorphism?

Answer. NO. It follows from the Löwenheim-Skolem-Tarski Theorem that if a first-order theory has infinite models then it has infinite models of any sufficiently large cardinality. Two models of different cardinality cannot be isomorphic.

32. has exactly two infinite models up to isomorphism?

Answer. NO. The reason is exactly the same as in the answer to the previous question.

Does the following hold?

33. For a set of first-order sentences, A, and a sentence φ, $A \models \varphi$ if and only if there is a finite subset B of A such that $B \models \varphi$.

Answer. YES. This is the Compactness Theorem, or, if you remember the Compactness Theorem in a different form, it follows from completeness.

34. A theory T is satisfiable if and only if it has a finite satisfiable subset.

Answer. NO. Then every theory would be satisfiable, since the empty set is satisfiable and is a subset of every theory. The right formulation is "...all of its finite subsets are satisfiable."

35. Every theory is a subset of some consistent theory.

Answer. NO. An inconsistent theory, no matter what you add to it, stays inconsistent.

36. Let φ be a sentence that has no finite models. Then $\neg\varphi$ has a countably infinite model.

Answer. YES. If φ is false in every finite model, then $\neg\varphi$ is true in every finite model, thus, by compactness, $\neg\varphi$ must be true in an infinite model.

Does the following hold?

37. Let T be an incomplete theory, over a countable language, with infinite models and no finite models. Then there are at least two non-isomorphic models of T for every infinite cardinality.

Answer. YES. Take a φ such that neither φ nor $\neg\varphi$ is a logical consequence of T. Let $\mathfrak{M}$ and $\mathfrak{N}$ be (countably infinite) models of $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ respectively. Then $\mathfrak{M}$ and $\mathfrak{N}$ are models of T, and they are not isomorphic.

38. There is a first-order sentence φ_{inf} such that, for every model $\mathfrak{M}$, $\mathfrak{M} \models \varphi_{\mathrm{inf}}$ if and only if $|\mathfrak{M}|$ is infinite.

Answer. NO. If there were such a sentence, then $\neg\varphi_{\mathrm{inf}}$ would hold exactly in all finite model; by compactness, it would have an infinite model, violating its own definition.

39. If $\sqsubset$ is well-founded over the set A, then every nonempty subset of $A \times A$ has a minimal element according to the relation $\prec$ defined by $(a, b) \prec (a', b') \Leftrightarrow a \sqsubset a'$ or $b \sqsubset b'$.

Answer. NO. For example, consider $A = \{0, 1\}$, $\sqsubset = \{(0, 1)\}$, and let the subset of $A \times A$ be $\{(0, 1), (1, 0)\}$.

Answer "yes" or "no". In the context of deductive tableaux:

40. Is there a first-order theory for which resolution alone, without skolemization, is complete for computing validity?

Answer. YES. For example, the inconsistent theory with axiom $\bot$.

41. Is there a first-order theory for which skolemization alone, without resolution, is complete for computing validity?

Answer. YES. For example, the inconsistent theory with axiom $\bot$.

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42. Let A be a set of first-order sentences and $\varphi[(\exists x\ \psi)^-] \in A$ (the minus sign denotes polarity). Let $A' = A \setminus \{\varphi[(\exists x\ \psi)^-]\} \cup \{\varphi[(\forall x\ \psi)^-]\}$. Is it true that if A is inconsistent, then A' is inconsistent?

Answer. NO. For a counterexample, take $A = \{\varphi\} = \{(\exists x\ x = a) \rightarrow \bot\}$. The reverse is true, that is, if A' is inconsistent then A is inconsistent. The reason is left as an exercise.

Suppose you are visiting a forest in which every inhabitant is either a knight or a knave. Knights always speak the truth and knaves always lie.

43. You witness the following conversation among three inhabitants A, B, and C:

A: At least one of the three of us is a knave.
B: C is a knight.

Do you now know for sure who among A, B, and C are the knights?

Answer. YES. A is the only knight: A cannot be a knave, because then he would have spoken the truth by saying at least one of the three was a knave. Thus A is a knight and she spoke the truth. Thus either B or C is a knave. If B is a knight, then C is a knight, too, thus B is a knave. And hence, since B is lying, C is a knave.

Alternatively, and more formally, if a stays for "A is a knight", $\neg a$ for "A is a knave", and so on, the conversation tells us that

$$a \leftrightarrow \neg a \vee \neg b \vee \neg c$$
$$b \leftrightarrow c$$

From this it is now provable, in propositional logic, that $a \wedge \neg b \wedge \neg c$.

44. Inspector Craig of Scotland Yard was called to the Forest of Knights and Knaves to help find a criminal named Arthur York. What made the process difficult was that it was not known whether Arthur York was a knight or a knave.

One suspect was arrested and brought to trial. Inspector Craig was the presiding Judge. Here is a transcript of the trial:

CRAIG: What do you know about Arthur York?

DEFENDANT: Arthur York once claimed that I was a knave.
CRAIG: Are you by any chance Arthur York?
DEFENDANT: Yes.

Is the defendant Arthur York?

Answer. NO. If the defendant is Arthur York, we get the following contradiction. Suppose he is Arthur York. Then he is a knight, since he claimed to be Arthur York. That would mean that his first answer to Craig was also true, which means that he, Arthur York, once claimed that he was a knave. But that is impossible! Therefore the defendant is not Arthur York, although he is, of course, a knave.

Alternatively, if a means "the defendant is Arthur York", k means "the defendant is a knight", and b means "Arthur York is a knight", we can formalize the information as

$$k \leftrightarrow (b \leftrightarrow \neg k) \text{ (first answer)}$$
$$k \leftrightarrow a \text{ (second answer)}$$
$$a \rightarrow (k \leftrightarrow b) \text{ (if he is AY then } k \text{ and } b \text{ equal)}$$

and from these prove $\neg a$.