Computer Science Department
Stanford University
Comprehensive Examination in Numerical Analysis
Fall 2001

Points for each question and its parts are specified
Books and notes are [redacted] permitted

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1 (10 points) Consider a linear least-squares data-fitting problem:

$$\min_{x \in R^n} \|Ax - b\|_2 \qquad (1),$$

A is a real $m \times n$ matrix, $m \geq n$, of full column rank, $b \in R^m$.

a) Describe the normal equations method for solving (1) (3pts)
b) Describe the Householder transformations method for solving (1) (4pts)
c) Compare the two abovementioned methods in terms of accuracy (relative error in x) and computational cost (number of flops) (3pts)

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2 (10 points) Consider the equation

$$y'(t) = \lambda y(t)$$
$$y(0) = 1, \qquad \lambda \in R, \quad \lambda < 0$$

a) (2pts)Give the exact analytical solution to the ODE and discuss the solution as $t \to +\infty$.

b) (3pts) Suppose we use the trapezoidal rule

$$y_{k+1} = y_k + \frac{\lambda y_k + \lambda y_{k+1}}{2} h,$$
$$y_0 = 1$$

Under what conditions will $y_k \to 0$ as $k \to \infty$?

c) (5pts) Now consider a general initial value problem $y' = f(t, y), \quad y(0) = 1$, and the trapezoidal rule for solving it:

$$y_{k+1} = y_k + \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2} h,$$
$$y_0 = 1$$

where f is sufficiently well-behaved. In particular it's Lipschitz-continuous in its 2nd argument with the constant L. Let $t_k = kh$. Obtain an estimate for

$$|\ y(t_k) - y_k\ |.$$

You can express the answer in terms of constants dependent on $y(t)$.

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3 (10 points) Numerical quadrature: Suppose we want to obtain an estimate for the integral of a function over an interval:

$$I(f) = \int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i), \qquad x_i \in [a,b], \quad x_1 < x_2 < \ldots < x_n$$

a) (3pts) Derive a two-point quadrature rule ($x_1 = a, \quad x_2 = b, \quad n = 2$) exact for polynomials of degree ≤ 1, e.g. by fitting a polynomial to the data points and integrating.

b) (4pts) Give an error estimate for this rule assuming that f(x) is sufficiently smooth.

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c) (3pts) To attain arbitrarily high accuracy in evaluating an integral we can subdivide the original interval into subintervals, apply a low-order quadrature rule in each subinterval, and sum the results. This is called a *composite* quadrature rule. For example, if the interval $[a,b]$ (we don't care if the ends belong to the interval or not) is partitioned into n subintervals $[x_{i+1}, x_i]$, $i = 1, \ldots, n$, $a = x_0 < x_1 < \ldots < x_n = b$, the composite trapezoid rule has the form

$$I(f) \approx \sum_{i=1}^{n} (w_1^{(i)} f(x_{i-1}) + w_2^{(i)} f(x_i))$$

Derive an error estimate for this rule. For simplicity take the subintervals to be of equal length.