Comprehensive Examinations in Logic
November 2001

30 questions

Time: 1 hour

Instructions

- The exam is open book and open notes. But no laptops or electronic accessories are allowed.
- Answer each question in the booklet itself. The answers should fit the space given. Writing on the margin/footer will **not** be considered.
- It is strongly recommended that you work out the answer outside the test booklet before attempting it.
- All questions have penalties for wrong answers. Read the instructions carefully before you start.
- Do not open the test booklet until instructed to do so.
- **THE HONOUR CODE:**
  1. The honor code is an undertaking of the students individually and collectively :
     (a) that they will not give or receive aid in examinations; they will not give or receive unauthorized aid in class work, in the preparation of reports, or in any other work that is to be used by the instructors as the basis of grading;
     (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the honor code.
  2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. the faculty will avoid, as far as possible, academic procedures that create temptation to violate the Honour code.
  3. While the faculty alone have the right and the obligation to set academic requirements, the students and the faculty will work together to establish optimal conditions for honorable academic work.

By writing my "magic–number" below, I acknowledge and accept the honor code.

WRITE MAGIC NUMBER: ____________

38

Section A

For each question in this section you need to choose one out the four possible choices provided. If you answer correctly you get **two points**. Answering incorrectly will result in **two points** being **deducted** from your score. Indicate correct answer by writing your choice clearly in the box provided. **No points deducted for leaving questions unanswered.**

1. A sentence ψ is valid **if and only if**

   (a) $(\neg\ \psi)$ is satisfiable.

   (b) ψ is satisfiable.

   (c) $(\neg\ \psi)$ is valid.

   (d) $(\neg\ \psi)$ is unsatisfiable

2. Using deductive tableaux for propositional logic with resolution and the **"polarity strategy"**. Which of the following statements is correct:

   (a) Given any propositional sentence ψ, we can prove ψ valid.

   (b) Given any propositional sentence ψ, we can prove ψ satisfiable.

   (c) If ψ is valid, we can find a proof using resolution and the polarity strategy.

   (d) Given an invalid sentence (ψ), we can prove $(\neg\ \psi)$ valid.

3. Which of the following statements is necessarily true about a well-founded relation $<$ over $A \times A$?

   (a) A is an infinite set.

   (b) $<$ is irreflexive.

   (c) $<$ is transitive.

   (d) A is a finite set.

39

4. Let Σ be a **set** of first-order sentences. Which of the following statements about Σ is necessarily true?

(a) If $\Sigma \models \psi$ and $\Sigma \models \neg \psi$ for some ψ then every interpretation is a model of Σ.

(b) If Σ is unsatisfiable then every subset of Σ is unsatisfiable.

(c) If $\Sigma \models \psi$ and $\Sigma \models \neg \psi$ for some ψ then Σ does not have a model.

(d) If Σ is empty then there is no ψ such that $\Sigma \models \psi$.

5. Let $\mathcal{F}$ be a valid sentence in predicate logic.

I $\mathcal{F}$ is also valid in the theory of natural numbers **with** 0,1,addition and multiplication.

II $\mathcal{F}$ is valid in the theory of natural numbers **with** 0,1, addition and **without** multiplication.

III There are **first–order** theories $\mathcal{T}$ such that $\mathcal{T} \not\models \mathcal{F}$.

Which of the statements above are necessarily true?

(a) II only

(b) I and II only

(c) I,II and III

(d) none of them.

6. Let $\mathcal{F}$ and $\mathcal{G}$ be sentences in propositional logic and let $\mathcal{H}$ be the sentence (*if* $\mathcal{F}$ *then* $\mathcal{G}$). If $\mathcal{G}$ is not satisfiable then which of the following is necessarily true?

(a) $\mathcal{H}$ is valid.

(b) If $\mathcal{H}$ is not valid then $\mathcal{F}$ is valid.

(c) The sentence ($\mathcal{H}$ *or* $\mathcal{F}$) is valid.

(d) None of the above.

40

7. Let $\mathcal{F}$ and $\mathcal{G}$ be sentences in first-order logic. Given that $\mathcal{F}$ is valid precisely when $\mathcal{G}$ is valid, which of the following statements are necessarily true?

   I $(\mathcal{F} \equiv \mathcal{G})$ is valid.

   II $(\mathcal{F} \equiv \mathcal{G})$ is satisfiable.

   (a) I only.
   (b) II only.
   (c) Both I and II.
   (d) neither I nor II.

   $\square$

8. Given a sentence $\mathcal{F}$, we obtain $\mathcal{G}$ by removing all its quantifiers through **validity preserving** skolemization. Which of the following statements is true about $\mathcal{F}$ and $\mathcal{G}$?

   (a) $\mathcal{G}$ is equivalent to $\mathcal{F}$.
   (b) $\mathcal{G}$ may not exist for any given $\mathcal{F}$.
   (c) It is always possible to skolemize in such a way that $\mathcal{G}$ is equivalent to $\mathcal{F}$.
   (d) If $\mathcal{F}$ and $\mathcal{G}$ are not equivalent then $\mathcal{F}$ is not valid.

   $\square$

9. You are given a first–order sentence ψ such that ψ has an infinite model. Which of the following are **necessarily** true

   I ψ has a countably infinite model.

   II ψ has a model of any infinite cardinality.

   III There is some infinite cardinality κ_0 such that ψ does not have a model of cardinality κ_0.

   (a) I and II only.
   (b) I and III only.
   (c) II only.
   (d) I only.

   $\square$

41

10. Which of the following statements is necessarily true about a set Σ of first-order sentences?

   (a) Consistency of Σ is decidable.

   (b) No finite subset of Σ is consistent.

   (c) If $\Sigma \models \psi$ then there is some finite subset Γ of Σ s.t. $\Gamma \models \psi$.

   (d) If $\Sigma \models \psi$, every subset of Σ entails ψ.

Section B

For each question in this section you need to write "yes" if you think the statement holds or "no" if you think it does not. You receive **two** points if you answer correctly. However **one** point will be deducted for wrong answers. There is no penalty for leaving questions unanswered.

1. If a sentence φ is satisfiable and does not have a finite model, $\neg\varphi$ is satisfiable.

2. If a sentence φ is not satisfiable in any finite model but has an infinite model, $\neg\varphi$ has a countably infinite model.

3. Any finitely axiomatizable first-order theory is recursive.

4. The transitive closure of a well-founded relation is itself well-founded.

5. We start with the empty set and for every first-order sentence ψ that is not equivalent to any sentence previously added to the theory, we add either ψ or $(\neg\psi)$. Any theory so formed is consistent.

42

6. The following terms are unifiable

$$h(x,\ f(y,a),\ y) \ \text{ and } \ h(f(f(b,\ z),\ z),\ x,\ f(z,a))$$

7. The result of removing all the quantifiers of the sentence $\mathcal{F}$ below by validity preserving skolemization is equivalent to $\mathcal{F}$.

$$\mathcal{F}: \ \ \textit{if} \ (\forall y)(\exists z)p(y,z) \ \ \textit{then} \ (\forall z')(\exists y')p(y',z')$$

8. Given a relation $R \subseteq A \times B$, we define its symmetric closure as the smallest symmetric relation R' containing R. It is possible for the symmetric closure of some well-founded relation over a non-empty domain to be well-founded.

9. In the deductive tableaux for propositional logic, given a sentence ψ, we write the sentence in the goal column as G_0. $G_{i+1},\ i \geq 0$ is obtained by applying $\mathcal{GG}$ resolving G_i with itself according to the polarity strategy. If ψ is valid, the procedure terminates with "true" as a final goal.

10. We can use the procedure above to prove a given sentence non-valid.

11. It is possible to write a first-order sentence that is false on all infinite models.

12. There is a finitely axiomatizable first-order theory whose only countably infinite model is the set of natural numbers, with 0, 1, addition, multiplication, less than relation.

43

13. If resolution is applied going against the "polarity strategy" one can prove a non-valid sentence.

14. The set of Gödel numbers of valid sentences in the language of Peano Arithmetic is recursive.

15. There are statements that can be proven using **Complete** Induction and not by **Stepwise** Induction.

16. $\{(\textit{ if } p \textit{ then } p_2), (\textit{ if } p \textit{ then } p_1), p\} \models (p_1 \textit{ or } p_2)$

17. Any first-order theory with equality can be converted to a theory without equality that has precisely the same models as the original theory, by adding a finite number of sentences that axiomatize equality.

18. Given any two models, we say that they are elementarily equivalent if and only if they satisfy the same first-order sentences. Is elementary equivalence of two finite models decidable?

19. Given a first-order theory Σ, we define $\mathcal{M}(\Sigma)$ as the set of models of Σ. Similarly given any set of models Φ, we define $\mathit{Th}(\Phi)$ as the set of First order sentences that are true in all the models of Φ. If Σ is complete and consistent, then $\mathit{Th}(\mathcal{M}(\Sigma)) = \Sigma$.

20. The propositional implication connective is complete (i.e., *and*, *or*, $\neg$ may be derived from implication).

44