Comprehensive Exam: Algorithms and Concrete Mathematics
Autumn 2001

This is a one hour closed-book exam and the point total for all questions is 60.
All of the intended answers may be written within the space provided. If necessary, you may use the back of the preceding page for additional scratch work. If you to use the back side of a page to write part of your answer, be sure to mark your answer clearly.
The following is a statement of the Stanford University Honor Code:

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

By writing my "magic number" below, I certify that I acknowledge and accept the Honor Code.

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(Number)

Prob	# 1	# 2	# 3	# 4	# 5	Total
Score						
Max	10	15	10	10	15	60

1. *(10 points)* ***Big-Oh notation, Running times***
For each of the following functions, circle all of the correct answers. Note that any single mistake (an incorrect answer marked or one of the correct answers not marked) will cost you all of the points for this question. No need to provide proofs.

   (a) *(3 points)* $f(n) = n^\epsilon, 0 < \epsilon \leq 1, \epsilon$ constant.
   Choose: $O(n)$ $O(n^2)$ $O(n^3)$ $O(\log n)$ $O(n \log n)$ $O(2^n)$ none

   $n = 2^k$
   $f(n) = 2^{k \frac{\log k}{k}}$
   $= 2^{\log k}$
   $= k$

   (b) *(3 points)* $f(n) = n^{\frac{\log \log n}{\log n}}$
   Choose: $o(\log n)$ $O(\log n)$ $\omega(\log n)$ $\Omega(\log n)$ $O(n)$ $O(n \log n)$ none

   (c) *(4 points)* $f(n) = \log^*(\log n)$
   Choose: $o(\log^*(\log^2 n))$ $\Theta(\log^* n)$ $O(\log^* n)$ $o(\log \sqrt{n})$ none

2. *(15 points)* ***Recurrences.***
Solve each of these three recurrences. You may give an exact solution, or give a good upper-bound using big-oh notation.

   (a) *(3 points)* $T(n) = T(\sqrt{n}) + n$, $T(2) = 1$.

   $m = \lg n$ $S(m) = T(2^m)$

   (b) *(4 points)* $T(n) = 5T(n/4) + \sqrt{n}$, $T(2) = 1$.

   (c) *(4 points)* $T(n) = 0.5(n^2)T(n/2)$, $T(2) = 1$.

   $\frac{n^2}{2} \cdot \frac{n^2}{2^2} \cdot \frac{n^2}{4^2} \cdot \frac{n^2}{8^2} \cdots 1$

   $\frac{(n^2)^{\log n}}{\sim} =$

   (d) *(4 points)* $T(n) = 2T(\frac{n}{3} + \log n) + n$, $T(1000) = 1$.
   ("1000" was chosen to emphasise that you should not be concerned with small values of n.)

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3. (*10 points*) You might recall that, in certain types of weighted graphs, Dijkstra's shortest path algorithm does not produce a correct shortest path. Present an example where Dijkstra's algorithm will not produce a correct answer even though a shortest path exists and it is unique. In other words, the algorithm will produce a wrong value of a distance between two points. Explain step-by-step the execution of the algorithm on your example and point out why it fails.

4. *(10 points)* Consider the following experiment: you have n balls and n bins. Balls are numbered 1 through n. First you put each ball into a bin chosen uniformly at random from 1 to n. Let X_i denote the number of balls in the ith bin. Prove that expectation of $\max_i X_i$ is bounded by $O(\log n)$.

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5. *(15 points)* You are given n items with associated weights $w_1, w_2, \ldots, w_n$ and costs $c_1, c_2, \ldots, c_n$. The costs are all integers drawn from range $[0 \ldots 1000]$. The goal is to choose a subset of items such that the total weight will not exceed a given value W while the total cost will be maximized. Explain a polynomial time algorithm to solve this problem. Prove that your algorithm is correct and compute its asymptotic running time.

Is your algorithm still polynomial if instead of $[0 \ldots 1000]$, the range is $[0 \ldots k]$, where k is part of the input? Explain.

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