ANSWER SHEET
Comprehensive Examination in LOGIC
November 2000

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THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code. (Signed) ____________________

Comprehensive Examination in Logic

Stanford University

Department of Computer Science

November 2000

Instructions

Please read these instructions and the *Notation* section carefully. Do not read beyond this page until instructed to do so.

You should mark your ansers only in the **answer sheet** that is provided with this part of the Comprehensive Examination. Be sure to write your **magic number** on the answer sheet.

This exam is **open book** and is composed of **42 questions** on **7 pages**, plus one answer sheet. For each question, write either **YES** or **NO** in the corresponding box of the answer sheet, or leave it blank. You will receive **+2** points for each correct answer, **−3** points for each incorrect answer, and **0** points for a blank (or crossed out) answer. You have **60 minutes** to complete the exam.

Notation

The notation is the one used by Enderton in *A Mathematical Introduction to Logic*, with the difference that the equality symbol is denoted by $=$ instead of $\approx$ and arguments to predicate and function symbols are enclosed in parentheses and separated by commas. Thus, for example, instead of Enderton's $fxyz$, $f(x, y, z)$ is used.

In some problems, the following symbols are used, whose definition is repeated here for completeness:

- $\mathrm{Cn}(A)$ is the set of consequences of an axiom set A;
- $\mathrm{Th}(\mathfrak{M})$ is the first-order theory of the structure $\mathfrak{M}$, i.e. the set of first-order sentences, of a given language, that are true in $\mathfrak{M}$.

Do not turn this page until instructed to do so.

Propositional Logic

Which of the following are complete sets of connectives?

1. $\wedge, \vee$
2. $\rightarrow, \neg$

If P means *"toves are slithy"* and Q means *"borogoves are mimsy"*, which of the following formulas mean *"toves are not slithy, unless borogoves are mimsy"*?

3. $\neg P \rightarrow Q$
4. $P \rightarrow Q$
5. $Q \rightarrow P$

Predicate Logic

Which of the following is a valid sentence of first-order logic?

6. $(\forall x\ P(x)) \rightarrow (\exists y\ P(y))$
7. $(\exists x\ P(x)) \rightarrow (\forall y\ P(y))$
8. $\exists x\ (P(x) \rightarrow \forall y\ P(y))$
9. $\forall x\ (\neg(x = 0) \rightarrow \exists y\ x = \mathrm{S}(y))$

Unification

Which of the following are true about unification in first-order logic?

10. $\{x \leftarrow z, y \leftarrow f(z)\}$ is an m.g.u. (most general unifier) of $f(g(x), y)$ and $f(g(y), f(x))$.
11. $\{y \leftarrow f(x)\}$ is an m.g.u. of $f(f(x), y)$ and $f(y, f(x))$.
12. $\{x \leftarrow y, y \leftarrow f(y)\}$ is an m.g.u. of $f(f(x), y)$ and $f(y, f(x))$.
13. Let θ be a unifier of t_1 and t_2. Then $\langle t_1, t_2, t_3 \rangle$ are unifiable if and only if $t_1\theta$ and $t_3\theta$ are unifiable.

Skolemization

In the following, "to skolemize" means to skolemize preserving validity, as in *The Deductive Foundations of Computer Programming.* x, x_1, x_2, y, z are variables.

14. Is the existential closure of $\neg P(x_1, y) \rightarrow \neg P(x_2, f(x_2))$ a correct skolemization of $(\neg\forall x\, \exists y\; P(x, y)) \rightarrow (\exists x\, \forall y\; \neg P(x, y))$?

15. Is the existential closure of $\neg(P(z, y) \rightarrow P(z, f(z)))$ a correct skolemization of $\exists y\; \neg\forall z\; (P(z, y) \rightarrow \exists y\; P(z, y))$?

Deductive Tableaux

Consider the following deductive tableau:

	A	G
1	$P(x, f(x)) \wedge Q(y) \rightarrow P(f(y), f(f(y)))$	
2	$P(f(a), f(f(a))) \vee Q(a)$	
3		$P(x, f(x))$

Which of the following rows can be added to the tableau by one correct application of a resolution rule?

16.

4	$\neg Q(a)$	

17.

4		$\neg Q(a)$

18.

4	$P(x, f(a)) \wedge Q(a)$	

19.

4		$P(x, f(a)) \wedge Q(a)$

Given a deductive tableau $\mathcal{T}$, for a first-order logic, in which there is no occurrence of the equality symbol, quantifiers, $\top$ or $\bot$, which of the following are true?

20. If an assertion and a goal in $\mathcal{T}$ are unifiable, then $\mathcal{T}$ is valid.

21. If no resolution rule can be applied, then $\mathcal{T}$ is not valid.

22. There exists a tableau containing only assertions (no goals) to which $\mathcal{T}$ is equivalent.

Let A be a finite set of axioms for a theory $T = \mathrm{Cn}(A)$, over a language with equality, φ a formula in the same language. Which of the following are then necessarily true?

23. If, starting from a tableau containing only formulas in A as assertions and only φ as goal, after a finite number of applications of resolution, quantifier elimination, and equality rules one gets a tableau with $\top$ as a goal or $\bot$ as an assertion, then $T \models \varphi$.

24. Let *SPO* be the theory of strict partial orderings (over the language with the binary predicate symbol $\prec$ and no equality) given by the two axioms *tr* (for *"transitivity"*) and *ir* (for *"irreflexivity"*), i.e. $SPO = \mathrm{Cn}(\{tr, ir\})$. Is it true, then, that a necessary and sufficient condition for $SPO \models \varphi$ is the existence of a tableau proof starting from the initial tableau

1.		$\neg tr \lor \neg ir \lor \varphi$

?

Polarity

Let A be a set of first-order sentences and A' be obtained from A by replacing every occurrence of a P-atom of positive polarity by $\top$. (A P-atom is an atomic formula of the form $P(\dots)$, where P is a predicate symbol of arbitrary arity.) Let $T = \mathrm{Cn}(A)$ and $T' = \mathrm{Cn}(A')$. Which of the following hold?

25. If $T \models \varphi$, then $T' \models \varphi$.

26. If $\mathfrak{M}$ is a model for T, then $\mathfrak{M}$ is a model for T'.

First-Order Theories

Which of the following are decidable?

27. The set of proofs in the language $(\mathbf{0}, \mathbf{S}, +, \cdot)$.

28. The set of sentences true in the structure $(\mathbb{N}, 0, S, +)$.

29. The set of sentences valid in the first-order logic of the language $(\mathbf{0}, \mathbf{S}, +, \cdot)$.

Which of the following are true? ($\mathfrak{N}$ is the structure $(\mathbb{N}, 0, S, <, +, \cdot, E)$, i.e. the standard model of natural numbers; E is the exponentiation function)

30. All countable models of $\mathrm{Th}(\mathfrak{N})$ are elementarily equivalent.

31. All countable models of $\mathrm{Th}(\mathfrak{N})$ are isomorphic.

32. $\mathrm{Th}(\mathfrak{N})$ is complete.

33. $\mathrm{Th}(\mathfrak{N})$ is recursively enumerable.

Consider a first-order language with one binary predicate symbol R and equality. Which of the following hold in this language?

34. There is a satisfiable formula all whose models are finite.

35. There is a satisfiable formula all whose models are infinite.

36. There is a satisfiable formula all whose models are countably infinite.

Consider a first-order language with equality, one binary predicate symbol R, and no other parameters. Furthermore, consider the theory $T = \mathrm{Cn}(A)$ of all logical consequences of axiom set A:

$$\forall x \forall y \forall z \ (x = y \lor y = z \lor x = z)$$
$$\forall x \ R(x, x)$$

37. Is T complete?

38. Is T decidable?

39. Is T recursively enumerable?

40. Is T axiomatizable?

Well-Founded Induction

Which of the following relations are well-founded?

41. On tuples (as defined in *The Deductive Foundations of Computer Programming* , the relation R defined by

$$R(x, y) \leftrightarrow x = tail(y) \ .$$

42. On non-negative integers, the relation R defined by

$$\begin{aligned} R(x, y) \leftrightarrow\ & \exists z \ (\ y = \mathrm{S}(\mathrm{S}(0)) \cdot z \ \wedge \ x + \mathrm{S}(0) = z \ ) \\ & \vee \exists z \ (\ (\mathrm{S}(\mathrm{S}(0)) \cdot z) + \mathrm{S}(0) = y \ \wedge \ x = z + \mathrm{S}(0) \ ) \ . \end{aligned}$$