COMPREHENSIVE EXAMINATION IN LOGIC
STANFORD UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE
NOVEMBER 2000

SOLUTIONS

# Instructions

Please read these instructions and the *Notation* section carefully. Do not read beyond this page until instructed to do so.

You should mark your ansers only in the **answer sheet** that is provided with this part of the Comprehensive Examination. Be sure to write your **magic number** on the answer sheet.

This exam is **open book** and is composed of **42 questions** on **12 pages**, plus one answer sheet. For each question, write either **YES** or **NO** in the corresponding box of the answer sheet, or leave it blank. You will receive **+2** points for each correct answer, **−3** points for each incorrect answer, and **0** points for a blank (or crossed out) answer. You have **60 minutes** to complete the exam.

# Notation

The notation is the one used by Enderton in *A Mathematical Introduction to Logic*, with the difference that the equality symbol is denoted by $=$ instead of $\approx$ and arguments to predicate and function symbols are enclosed in parentheses and separated by commas. Thus, for example, instead of Enderton's $fxyz$, $f(x, y, z)$ is used.

In some problems, the following symbols are used, whose definition is repeated here for completeness:

- $\mathrm{Cn}(A)$ is the set of consequences of an axiom set $A$;
- $\mathrm{Th}(\mathfrak{M})$ is the first-order theory of the structure $\mathfrak{M}$, i.e. the set of first-order sentences, of a given language, that are true in $\mathfrak{M}$.

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**Do not turn this page until instructed to do so.**

# Propositional Logic

Which of the following are complete sets of connectives?

**1.** $\wedge, \vee$

**Answer.** NO. It can be proved by induction on formulas build from these connectives that every such formula is true under the truth assignment that assigns a true value to every propositional symbol. Thus, for example, no contradiction can be written in this language.

**2.** $\rightarrow, \neg$

**Answer.** YES. $\varphi \vee \psi$ is equivalent to $\neg\varphi \rightarrow \psi$, $\varphi \wedge \psi$ is equivalent to $\neg(\varphi \rightarrow \neg\psi)$, etc.

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If $P$ means *"toves are slithy"* and $Q$ means *"borogoves are mimsy"*, which of the following formulas mean *"toves are not slithy, unless borogoves are mimsy"*?

**3.** $\neg P \rightarrow Q$

**Answer.** NO.

**4.** $P \rightarrow Q$

**Answer.** YES.

**5.** $Q \rightarrow P$

**Answer.** NO.

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# Predicate Logic

Which of the following is a valid sentence of first-order logic?

6. $(\forall x\ P(x)) \rightarrow (\exists y\ P(y))$

**Answer.** YES. This depends on the assumption that the domain of a first-order interpretation cannot be the empty set. Alternatively, the formula can be proved in your favorite calculus.

7. $(\exists x\ P(x)) \rightarrow (\forall y\ P(y))$

**Answer.** NO. As a counterexample, take an interpretation with a two-element domain, $P$ holding of one only.

8. $\exists x\ (P(x) \rightarrow \forall y\ P(y))$

**Answer.** YES. This is known as *Beth's formula*. Given any interpretation, there are two cases: either $\forall y\ P(y)$ is true, and then the implication is always true and any value for $x$ will do, or it is false, and then there is an element of the domain for which $P$ does not hold, and one can assign that element to $x$, making the antecedent of the implication always false, hence the whole implication true.

9. $\forall x\ (\neg(x = 0) \rightarrow \exists y\ x = \mathbf{S}(y))$

**Answer.** NO. This is valid in models of arithmetic, but is not a validity of first-order logic. For a counterexample, take a two-element domain $\{a, b\}$, map $\mathbf{0}$ to $a$, let $\mathbf{S}$ be interpreted as the identity function returning $a$.

# UNIFICATION

Which of the following are true about unification in first-order logic?

**10.** $\{x \leftarrow z, y \leftarrow f(z)\}$ is an m.g.u. (most general unifier) of $f(g(x), y)$ and $f(g(y), f(x))$.

**Answer.** NO. The two terms are not unifiable. To see this, apply the unification algorithm and you will end with an occur-check violation (an equation of the form $x = t$, with $x$ occurring in $t$).

**11.** $\{y \leftarrow f(x)\}$ is an m.g.u. of $f(f(x), y)$ and $f(y, f(x))$.

**Answer.** YES. It is one of the possible solutions returned by the unification algorithm.

**12.** $\{x \leftarrow y, y \leftarrow f(y)\}$ is an m.g.u. of $f(f(x), y)$ and $f(y, f(x))$.

**Answer.** YES. All m.g.u.s of a term differ by a permutation of variables, and this can be obtained by the one in the previous problem by composition with the permutation $\{x \leftarrow y, y \leftarrow x\}$. Note that this m.g.u. cannot be returned by any execution of the unification algorithm, because it is not idempotent.

**13.** Let $\theta$ be a unifier of $t_1$ and $t_2$. Then $\langle t_1, t_2, t_3\rangle$ are unifiable if and only if $t_1\theta$ and $t_3\theta$ are unifiable.

**Answer.** NO. $\theta$ needs to be most general for this to hold. Consider $t_1 = f(x)$, $t_2 = f(y)$, $t_3 = f(g(z))$, $\theta = \{x \leftarrow f(x), y \leftarrow f(x)\}$. $\theta$ is not most general and $\langle t_1\theta, t_3\theta\rangle$ is not unifiable, but $\langle t_1, t_2, t_3\rangle$ clearly is.

## Skolemization

In the following, "to skolemize" means to skolemize preserving validity, as in *The Deductive Foundations of Computer Programming.* $x$, $x_1$, $x_2$, $y$, $z$ are variables.

**14.** Is the existential closure of $\neg P(x_1, y) \to \neg P(x_2, f(x_2))$ a correct skolemization of $(\neg\forall x\, \exists y\; P(x,y)) \to (\exists x\, \forall y\; \neg P(x,y))$?

**Answer.** NO. The occurrence of $x_1$ should be replaced by a constant.

**15.** Is the existential closure of $\neg(P(z,y) \to P(z, f(z)))$ a correct skolemization of $\exists y\; \neg\forall z\; (P(z,y) \to \exists y\; P(z,y))$?

**Answer.** NO. A correct skolemization would be
$\neg(P(z,y) \to P(z, f(y,z)))$.

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## Deductive Tableaux

Consider the following deductive tableau:

| | A | G |
|---|---|---|
| 1 | $P(x, f(x)) \wedge Q(y) \to P(f(y), f(f(y)))$ | |
| 2 | $P(f(a), f(f(a))) \vee Q(a)$ | |
| 3 | | $P(x, f(x))$ |

Which of the following rows can be added to the tableau by one correct application of a resolution rule?

**16.**

| | | |
|---|---|---|
| 4 | $\neg Q(a)$ | |

**Answer.** NO.

**17.**

| | | |
|---|---|---|
| 4 | | $\neg Q(a)$ |

**Answer.** YES. Resolve 2 and 3 according to the polarity strategy.

**18.**

| | | |
|---|---|---|
| 4 | $P(x, f(a)) \wedge Q(a)$ | |

**Answer.** NO.

**19.**

| | | |
|---|---|---|
| 4 | | $P(x, f(a)) \wedge Q(a)$ |

**Answer.** NO.

The complete set of rows that can be inferred from the given ones by one application of a resolution rule is: goals $\bot$, $P(x, f(x)) \wedge Q(y)$, $\neg Q(a)$ and assertions $\top$, $(Q(y) \rightarrow P(f(y), f(f(y)))) \vee Q(a)$, $(P(x, f(x)) \rightarrow P(f(a), f(f(a)))) \vee P(f(a), f(f(a)))$, $\neg(P(x, f(x)) \wedge Q(a)) \vee Q(a)$.

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Given a deductive tableau $\mathcal{T}$, for a first-order logic, in which there is no occurrence of the equality symbol, quantifiers, $\top$ or $\bot$, which of the following are true?

**20.** If an assertion and a goal in $\mathcal{T}$ are unifiable, then $\mathcal{T}$ is valid.

**Answer.** YES. A resolution step can be applied and yield the true goal in one step.

**21.** If no resolution rule can be applied, then $\mathcal{T}$ is not valid.

**Answer.** YES. With no quantifiers there is no need for skolemization, and with no equality the resolution rule alone is complete for validity.

**22.** There exists a tableau containing only assertions (no goals) to which $\mathcal{T}$ is equivalent.

**Answer.** YES. Just move all goals in $\mathcal{T}$ to assertion by prepending a $\neg$. According to the Duality Proposition, the two tableaux are equivalent.

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Let $A$ be a finite set of axioms for a theory $T = \mathrm{Cn}(A)$, over a language with equality, $\varphi$ a formula in the same language. Which of the following are then necessarily true?

**23.** If, starting from a tableau containing only formulas in $A$ as assertions and only $\varphi$ as goal, after a finite number of applications of resolution, quantifier elimination, and equality rules one gets a tableau with $\top$ as a goal or $\bot$ as an assertion, then $T \models \varphi$.

**Answer.** YES. This is indeed a sufficient condition for validity. It is not necessary, though, since in general one must add the assertion $x = x$ to have completeness in presence of equality.

**24.** Let *SPO* be the theory of strict partial orderings (over the language with the binary predicate symbol $\prec$ and no equality) given by the two axioms *tr* (for *"transitivity"*) and *ir* (for *"irreflexivity"*), i.e. $SPO = \mathrm{Cn}(\{tr, ir\})$. Is it true, then, that a necessary and sufficient condition for $SPO \models \varphi$ is the existence of a tableau proof starting from the initial tableau

| | | |
|---|---|---|
| 1. | | $\neg tr \lor \neg ir \lor \varphi$ |

?

**Answer.** YES. The formula given as a goal is equivalent to $\psi$: $tr \land ir \rightarrow \varphi$. The existence of a tableau proof starting from here is equivalent to the validity of *psi*, which in turn is equivalent to the validity of $\varphi$ in *SPO*.

# Polarity

Let $A$ be a set of first-order sentences and $A'$ be obtained from $A$ by replacing every occurrence of a $P$-atom of positive polarity by $\top$. (A $P$-atom is an atomic formula of the form $P(\dots)$, where $P$ is a predicate symbol of arbitrary arity.) Let $T = \mathrm{Cn}(A)$ and $T' = \mathrm{Cn}(A')$. Which of the following hold?

**25.** If $T \models \varphi$, then $T' \models \varphi$.

**Answer.** NO. For a simple counterexample, take $\varphi$ to be $\forall x\ P(x)$ and $A$ to be $\{\varphi\}$. The converse implication is true, as can be inferred from the next problem.

**26.** If $\mathfrak{M}$ is a model for $T$, then $\mathfrak{M}$ is a model for $T'$.

**Answer.** YES. For every axiom $\varphi \in A$ let $\varphi'$ be the corresponding axiom in $A'$. Then, by the Polarity Proposition, $phi \rightarrow \varphi'$. For $\mathfrak{M}$ to be a model of $T$ means, by definition, that $\mathfrak{M} \models \varphi$ for all $\varphi in A$ and hence by the previous observation it follows that $\mathfrak{M} \models \varphi'$. Again by definition of being a model this means that $\mathfrak{M}$ is a model of $T'$.

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# First-Order Theories

Which of the following are decidable?

**27.** The set of proofs in the language $(\mathbf{0}, \mathbf{S}, +, \cdot)$.

**Answer.** YES. It can be checked algorithmically whether a given syntactic object is a correct proof. In general, any reasonable definition of proof must have this property.

**28.** The set of sentences true in the structure $(\mathbb{N}, 0, S, +)$.

**Answer.** YES. This is a substructure of Presburger Arithmetic.

**29.** The set of sentences valid in the first-order logic of the language $(\mathbf{0}, \mathbf{S}, +, \cdot)$.

**Answer.** NO. This is known as Church's Theorem.

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Which of the following are true? ($\mathfrak{N}$ is the structure $(\mathbb{N}, 0, S, <, +, \cdot, E)$, i.e. the standard model of natural numbers; $E$ is the exponentiation function)

**30.** All countable models of $\mathrm{Th}(\mathfrak{N})$ are elementarily equivalent.

**Answer.** YES. This holds in general for any structure $\mathfrak{M}$, since then $\mathrm{Th}(\mathfrak{N})$ is complete.

**31.** All countable models of $\mathrm{Th}(\mathfrak{N})$ are isomorphic.

**Answer.** NO. Add a new constant $c$ to the language, and add to $\mathrm{Th}(\mathfrak{N})$ the axioms $\mathbf{0} < c$, $\mathbf{S}(\mathbf{0}) < c$, $\mathbf{S}(\mathbf{S}(\mathbf{0})) < c$, etc. The resulting theory is finitely satisfiable, hence satisfiable by the Compactness Theorem. The restriction of a model of it to the language without $c$ is a model of $\mathrm{Th}(\mathfrak{N})$ that is not isomorphic to $\mathfrak{N}$.

**32.** $\mathrm{Th}(\mathfrak{N})$ is complete.

**Answer.** YES. This holds for any structure $\mathrm{Th}(\mathfrak{N})$.

**33.** $\mathrm{Th}(\mathfrak{N})$ is recursively enumerable.

**Answer.** NO. This theory is undecidable.

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Consider a first-order language with one binary predicate symbol $R$ and equality. Which of the following hold in this language?

**34.** There is a satisfiable formula all whose models are finite.

**Answer.** YES. Take $\forall x\ x = x$.

**35.** There is a satisfiable formula all whose models are infinite.

**Answer.** YES. The standard example is

$$\begin{gathered}\forall x\ \neg R(x,x) \land \\ \forall x \forall y \forall z\ (R(x,y) \land R(y,z) \rightarrow R(x,z)) \land \\ \forall x\, \exists y\ R(x,y)\ .\end{gathered}$$

**36.** There is a satisfiable formula all whose models are countably infinite.

**Answer.** NO. This would violate the Löwenheim-Skolem Theorem.

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Consider a first-order language with equality, one binary predicate symbol $R$, and no other parameters. Furthermore, consider the theory $T = \mathrm{Cn}(A)$ of all logical consequences of axiom set $A$:

$$\forall x \forall y \forall z\ (x = y \lor y = z \lor x = z)$$
$$\forall x\ R(x, x)$$

**37.** Is $T$ complete?

**Answer.** NO. As a counterexample, neither the sentence $\forall x \forall y\ (R(x, y) \rightarrow R(y, x))$ nor its negation are valid in the theory. To reason about this problem, you should realize that the models can be thought as being the graphs with at most three nodes and having reflexive loops.

**38.** Is $T$ decidable?

**Answer.** YES. One simply has to check a finite set of graphs and see whether a given sentence holds in all of them.

**39.** Is $T$ recursively enumerable?

**Answer.** YES. Any theory with a recursively enumerable set of axioms is recursively enumerable.

**40.** Is $T$ axiomatizable?

**Answer.** YES. A (finite, hence recursive) set of axioms is given by the problem.

# Well-Founded Induction

Which of the following relations are well-founded?

**41.** On tuples (as defined in *The Deductive Foundations of Computer Programming* , the relation $R$ defined by

$$R(x, y) \leftrightarrow x = tail(y) \ .$$

**Answer.** YES. If $R(x, y)$, then $x$ is a strictly shorter tuple than $y$,and hence there cannot be an infinite descending chain.

**42.** On non-negative integers, the relation $R$ defined by

$$\begin{aligned} R(x, y) \leftrightarrow & \ \exists z \ ( \ y = \mathrm{S}(\mathrm{S}(0)) \cdot z \ \wedge \ x + \mathrm{S}(0) = z \ ) \\ & \vee \exists z \ ( \ (\mathrm{S}(\mathrm{S}(0)) \cdot z) + \mathrm{S}(0) = y \ \wedge \ x = z + \mathrm{S}(0) \ ) \ . \end{aligned}$$

**Answer.** NO. In fact, this relation has a reflexive loop at 1. The whole point here was to see if you are acquainted with the language of first-order logic and can decrypt it fast enough.

# ANSWER SHEET
## Comprehensive Examination in LOGIC
## November 2000

MAGIC NUMBER: ________

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | NO | 2 | YES | 3 | NO | 4 | YES | 5 | NO |
| 6 | YES | 7 | NO | 8 | YES | 9 | NO | 10 | NO |
| 11 | YES | 12 | YES | 13 | NO | 14 | NO | 15 | NO |
| 16 | NO | 17 | YES | 18 | NO | 19 | NO | 20 | YES |
| 21 | YES | 22 | YES | 23 | YES | 24 | YES | 25 | NO |
| 26 | YES | 27 | YES | 28 | YES | 29 | NO | 30 | YES |
| 31 | NO | 32 | YES | 33 | NO | 34 | YES | 35 | YES |
| 36 | NO | 37 | NO | 38 | YES | 39 | YES | 40 | YES |
| 41 | YES | 42 | NO | | | | | | |

THE STANFORD UNIVERSITY HONOR CODE

A. The Honor Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

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