Comprehensive Exam Solutions
Automata and Formal Languages (30 points)
Autumn 1996

1. [10] Give context-free grammars generating the following languages over the alphabet $\{a, b\}$. (a) $\{a^i b^j a^{i+j+k} b^k \mid i, j, k \geq 0\}$;

$$
\begin{aligned}
S &\rightarrow AC && (1)\\
A &\rightarrow aAa && (2)\\
A &\rightarrow B && (3)\\
B &\rightarrow bBa && (4)\\
B &\rightarrow \epsilon && (5)\\
C &\rightarrow aCb && (6)\\
C &\rightarrow \epsilon && (7)
\end{aligned}
$$

(b) All strings with an equal number of a's and b's.

$$
\begin{aligned}
S &\rightarrow aSbS && (8)\\
S &\rightarrow bSaS && (9)\\
S &\rightarrow \epsilon && (10)
\end{aligned}
$$

2. [10] Consider the following decision problem. Given a regular expression over the alphabet $\{a, b\}$, decide whether the language it denotes contains a string of palindromes. (A string of palindromes is a member of P^* where P is the set of all palindromes.) State whether this problem is decidable, and briefly sketch the reason.

Decidable. The problem is equivalent to whether the given regular expression has a nonempty intersection with the set of strings of palindromes,

(16)

a context-free language. Given a regular language and a context-free language, their intersection is context-free and one may obtain a context-free grammar generating it and then test emptiness of the language so generated.

3. [10] For each of the following sets, state whether it is in NP, coNP, or neither. (If it is both you need not say so, in which case either NP or coNP is a correct answer.) Give a one-sentence reason for each.

(a) The set of three-colorable undirected planar graphs. (A three-coloring of a graph is an assignment of at most three distinct colors to the vertices of the graph such that every edge has different colors at each end.)

In NP, with membership witnessed by a planar embedding and a 3-coloring, checkable in linear time.

(b) The set of finite sets of signed binary integers having no subset adding to zero.

In coNP, with nonmembership witnessed by a subset adding to zero, checkable in linear time.

(c) (The knapsack problem). The set of pairs (w, c) where w is a finite list of positive binary numbers (items available to go in the knapsack) and c is an integer (the size of the knapsack) such that some subset of the elements of w sum to c (a perfect packing of the knapsack).

In NP, with membership witnessed by a subset of w summing to c, checkable in linear time.

(d) The set of pairs (i, j) of nonnegative integers written in binary such that $\varphi(i, j)$ holds but $\varphi(i', j)$ does not hold for $0 \leq i' < i$, where the predicate φ is computable in time polynomial in the length of its inputs written in binary.

In coNP, with nonmembership witnessed by i, i', j such that $i' < i$ and at least one of $\varphi(i, j)$ or $\varphi(i', j)$ is false.

(e) The set of valid computations of a universal Turing machine.

In P, hence in both NP and coNP.