Computer Science Department
Stanford University
Comprehensive Examination in Automata, Languages, and Logic
Autumn 1993

October 29, 1993

READ THIS FIRST!

1. You should write your answers for this part of the Comprehensive Examination in BLUE BOOKS. There are three problems in the exam; use a SEPARATE blue book for each problem. Be sure to write your MAGIC NUMBER on the cover of every blue book that you use.

2. The number of POINTS for each problem indicates how elaborate an answer is expected. The total number of points is 60, and the exam takes 60 minutes. This "coincidence" can help you plan your time.

4. This exam is OPEN BOOK. You may use notes, articles, or books—but no help from other sentient agents such as other humans or robots.

5. Show your work, since PARTIAL CREDIT will be given for incomplete answers.

6. there are 3 pages of the exam.

29

Proof:

(a) 8 points

Use the pumping lemma to show that $\overline{L}$, the complement of L, is not regular. By Theorem 3.2 of Hopcroft and Ullman, it follows that L cannot be regular.

For any n, let $z = 0^n 1^n$. Observe that $z \in \overline{L}$, i.e., z has the same number of 0's and 1's. For any u, v, w such that $z = uvw$ and $|uv| \leq n$, observe that v must consist only of 0's. Clearly $uv^iw \notin \overline{L}$ for any $i > 1$, so $\overline{L}$ is not regular.

(b) 12 points

Suppose L is recursive, i.e., there is a TM M_L that accepts L and halts on all inputs. Use M_L to construct a TM to decide L_u (the universal language) as follows. Given an input $\langle M, w \rangle$, construct a TM M' from M by adding an automaton state q, such that every transition to an accepting state in M instead goes to q. Observe that M' enters q on w if and only if M accepts w. It follows that L cannot be recursive.

2

Proof:

(a) (10 points)

Either of the following solutions is correct.

$$\mathcal{F}': \quad \begin{array}{l} \textit{if} \left(\begin{array}{l} \textit{if } p(w) \textit{ then } q(w, f(x,y,w)) \\ \quad \wedge \\ \textit{if } q(w,z) \textit{ then } p(w) \end{array} \right) \\ \textit{then } r(x,y) \end{array}$$

$$\mathcal{F}': \quad \begin{array}{l} \textit{if} \left(\begin{array}{l} p(w) \wedge q(w, f(x,y,w)) \\ \quad \vee \\ \textit{not } p(w) \wedge \textit{not } q(w,z) \end{array} \right) \\ \textit{then } r(x,y) \end{array}$$

(b) (10 points)

$\mathcal{F}: \quad (\forall x')q(f(g(x')), g(x)) \ \wedge \ \neg q(f(a), g(x))$

3 **Proof:**

(a) 15 points

If N is definable in $\mathfrak{N}_A$, then N is eventually periodic. Choose M and p such that, for every $n > M$, $n \in N$ if and only if $n + p \in N$. Construct a finite automaton with states indexed $s = 0, 1, \ldots, M + p + 1$ such that there is a transition from $s = i$ to $s' = i + 1$ for all $0 \leq i \leq M + p$, and a transition from $s = M + p + 1$ to $s' = M + 1$. Each state $s = i$ is an accepting state if and only if $i \in N$.

Conversely, assume that the unary representation L of N is accepted by some finite state automaton. Construct a DFA $(Q, \Sigma, \delta, q_0, F)$ accepting L, and assume that every $q \in Q$ is reachable from the initial state q_0. Since Σ is a singleton set (containing $\mathbf{0}$), there can only be a single cycle in the DFA. There is an obvious mapping I from Q to the set $\{0, 1, \ldots, |Q| - 1\}$, where $I(q_0) = 0$ and $I(q') = I(q) + 1$ if $\delta(q) = q'$ (except the "last state" q_l such that $I(q_l) = |Q| - 1$). Take $M = I(\delta(q_l)) - 1$ and $p = |Q| - I(q_l)$. Clearly $\mathbf{0}^n \in L$ if and only if $\mathbf{0}^{n+p} \in L$ for every $n > M$, implying that N is eventually periodic, and therefore definable in $\mathfrak{N}_A$.

(b) 5 points

By the Gödel Incompleteness Theorem and Theorems 33E and 34A of Enderton, there are relations definable in $\mathfrak{N}$ that are not recursive.