Computer Science Department
Stanford University
Comprehensive Examination in Analysis of Algorithms
Autumn 1992

October 19, 1992

You should write your answers for this part of the Comprehensive Examination in BLUE BOOKs. There are 3 problems in the exam.

Be sure to write your MAGIC NUMBER on the cover of every blue book that you use.

Show your work, since PARTIAL CREDIT will be given for incomplete answers.

COMPS and SOLUTIONS

1992

AA PROBLEMS

Note: The exam is open book. One hour

20 points each question

Problem 1. Given two points on the plane, $a = (x_1, y_1)$ and $b = (x_2, y_2)$, the *Manhattan distance* between a and b is given by $d(a,b) = |x_1 - x_2| + |y_1 - y_2|$.

Suppose you are given n points, $p_1, \ldots, p_n$, on the plane. and you want to find a point q such that

$$\sum_{1 \le i \le n} d(p_i, q)$$

is minimized. (The desired point does not have to be one of the input points.)

Give an algorithm to solve this problem. Explain why your algorithm is correct. Your algorithm should be as fast as possible.

Problem 2. Let $G = (V, E)$ be an undirected graph. Given $V' \subseteq V$, a *subgraph induced by* V' is (V', E') where $E' = \{\{v, w\} \in E | v, w \in V'\}$. *A coloring* of G is a labeling of V by integers (colors) such that for every edge, its endpoints have different colors.

Suppose a graph G has the property that every subgraph induced by a subset of nodes has the average degree below 7. (The average degree is the sum of node degrees divided by the number of nodes.) Give a polynomial-time algorithm to color G in 7 colors. (For this problem, you do not have to give the fastest algorithm to get full credit.)

Problem 3. How many n-digit binary sequences that have no adjacent 0's are there?

Comprehensive Examination — Autumn 1992

Automata, Formal Languages and Logic (60 points)

Please solve only TWO problems out of Problems 1, 2 and 3. If you solve all three, we will consider only the best two scores. You must also solve Problems 4 and 5.

Problem 1 (10 points)

Show that

$$\left\{ \begin{array}{l} \forall x.\ [\forall y.\ q(x,y) \rightarrow p(x)], \\ \neg p(a) \end{array} \right\} \vdash \neg \forall y.\ q(a,y)$$

where p and q are unary and binary predicate symbols, respectively. You may use *modus ponens* and axioms from the following four groups:

1. Tautologies;
2. $\forall x.\ \alpha \rightarrow \alpha^x_t$, where t is substitutable for x in α;
3. $\forall x.\ (\alpha \rightarrow \beta) \rightarrow (\forall x.\ \alpha \rightarrow \forall x.\ \beta)$;
4. $\alpha \rightarrow \forall x.\ \alpha$, where x does not occur free in α.

To receive full credit, indicate for each line of the deduction either the axiom group to which it belongs or the two earlier lines to which you are applying ***modus ponens***.

Problem 2 (10 points)

Skolemize the following sentence, and state whether the resulting formula preserves satisfiability or validity.

$$\forall x. \exists y. (\exists z.\ p(x,y,z) \rightarrow q(a,x,y))$$

Problem 3 (10 points)

Let $\prec_1$ and $\prec_2$ be binary relations over some domain, and define

$$x \prec_\vee y \quad \equiv \quad x \prec_1 y \vee x \prec_2 y.$$

Determine which (if any) of the following statements is true. Justify your answer.

a. If $\prec_1$ and $\prec_2$ are well-founded, then $\prec_\vee$ is well-founded.
b. If $\prec_\vee$ is well-founded, then $\prec_1$ and $\prec_2$ are well-founded.

Problem 4 (20 points)

Are the following problems decidable or not? Prove your answer.

a. Given a Turing machine M (with a semi-infinite tape) and a specific input string w, does M ever scan a tape cell more than once when started on input w?
b. Given a Turing machine M (with a semi-infinite tape) and a specific input string w, does M ever scan the LEFTMOST tape cell more than once when started on input w?

Problem 5 (20 points)

Reduce (directed or undirected, as you prefer) HAMILTONIAN CYCLE to SAT. A clear outline of the (polynomial time) reduction will suffice.

Comprehensive: Numerical Analysis (30 points). Autumn 1992

(Problem I)

(16 points). **Rootfinding for Nonlinear Equations**

(a) Define the *order of convergence* of a sequence $\{x_n | n \geq 0\}$ to a point α. When is the convergence said to be *linear*? If the convergence is linear, define the *rate of linear convergence*.

(b) Given a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ and two points $a, b : f(a)f(b) < 0$ define the *bisection method*, in algorithmic form, to find a root α in $[a, b]$ satisfying $f(\alpha) = 0$.

Let $c_1 = [a + b]/2$ and let $\{c_n | n \geq 1\}$ be the sequence of approximations to α generated by the method. Show that, at each step of the iteration, c_n is the mid-point of an interval I_n in which α is guaranteed to lie and whose length L_n satasfies

$$L_n = \frac{|b - a|}{2^{n-1}}.$$

Deduce that

$$|\alpha - c_n| \leq \frac{|b - a|}{2^n}.$$

What is the rate of convergence in this case?

(c) Given a twice continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ define Newton iteration to generate a sequence $\{x_n | n \geq 0\}$ to locate a root α satisfying $f(\alpha) = 0$.

Assume that $|f''(x)| \leq 2$ and $|f'(x)| \geq 1$ for all real x. By Taylor expansion show that

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2}{2} \frac{f''(\xi_n)}{f'(x_n)}$$

for some $\xi_n \in \mathbb{R}$. Prove that, if $|x_0 - \alpha| < 1$, then x_n converges to α with order 2.

(Problem II)

(14 points). **Linear Systems and Matrices**

(a) Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 - 3x_2 &= .4 \end{aligned}$$

Write this in the form $Ax = b$ where $x = (x_1, x_2, x_3)^T$ and the ordering of the equations is retained. Show that the LU factorization of A does not exist.

(b) Give a permutation B of the matrix A for which the LU factorization does exist. Calculate this factorization.

(c) Define the condition number K of a matrix A. If

$$Ax = b, \quad Ay = b + r$$

state the two basic inequalities which relate the relative error $\frac{\|x-y\|}{\|x\|}$ to the relative peturbation $\frac{\|r\|}{\|b\|}$. *Briefly* discuss the implications of these inequalities for the numerical solution of linear systems.

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SOLUTION: The Newton algorithm is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Now

$$0 = f(\alpha) = f(x_n + \alpha - x_n) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi_n)$$

for some real ξ_n. Solving for α gives

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2}{2} \frac{f''(\xi_n)}{f'(x_n)}.$$

Hence, substracting from the Newton iteration scheme we get

$$|x_{n+1} - \alpha| = \frac{|x_n - \alpha|^2}{2} \frac{|f''(\xi_n)|}{|f'(x_n)|}.$$

Now, we have that

$$|f''(x)| \leq 2, \quad \frac{1}{|f'(x)|} \leq 1$$

for all real x. Thus

$$|x_{n+1} - \alpha| \leq |x_n - \alpha|^2.$$

Thus

$$|x_n - \alpha| \leq |x_0 - \alpha|^{2^n}.$$

Quadratic convergence of x_n to α follows if $|x_0 - \alpha| < 1$.

(Problem II)

(14 points). **Linear Systems and Matrices**

(a) Consider the linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 1 \\ x_1 & - & 3x_2 & & & = & 4 \end{array}$$

Write this in the form $Ax = b$ where $x = (x_1, x_2, x_3)^T$ and the ordering of the equations is retained. Show that the LU factorization of A does not exist.

SOLUTION: We take

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & -3 & 0 \end{pmatrix}$$

and $b = (0, 1, 4)^T$.

Since

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

is a submatrix of A and is singular, the LU fatorisation does not exist.