Comprehensive Solutions: **Autumn 1991**

Automata, Languages, and Mathematical Theory of Computation (60 points)

Problem 1 (24 points). [2 pts for correct yes/no answer; 2 pts for correct reason].

1a. Yes. Easily established by induction on proofs (i.e., last rule used).

1b. No. $P1$ is $x := 0$; $P2$ is *while* $x > 0$ *do* $x := x - 1$.

1c. Yes. Use Rule 16.

1d. Yes. Use Rule 16

1e. No.

$y := y + 1$

$\vdash \{y = 0\}\ P2\ \{y = 1\}$

$\neg(\vdash \{y = 0\}\ P1\ \{y = 1\})$

1f. Yes. (Modulo missing "end" — a typo). Again use Rule 16.

Problem 2 (16 points).

2a. Axioms added include the following.

```
(OR (EQUAL (OTP X) T) (EQUAL (OTP X) F))
(OTP (OT X1 X2 X3))
(OTP (ET))
NOT (EQUAL (OT X1 X2 X3) (ET)))
(IMPLIES (AND (OTP X) (NOT (EQUAL X (ET))))
         (EQUAL (OT (LT X) (LABEL X) (RT X)) X))
(IMPLIES (OTP X1) (EQUAL (LT (OT X1 X2 X3)) X1))
(IMPLIES (NUMBERP X2) (EQUAL (LABEL (OT X1 X2 X3)) X2))
(IMPLIES (OTP X3) (EQUAL (RT (OT X1 X2 X3)) X3))
(IMPLIES (OR (NOT (OTP X))
             (EQUAL X (ET))
             (AND (NOT (OTP X1)) (EQUAL X (OT X1 X2 X3))))
         (EQUAL (LT X) (ET)))
(IMPLIES (OR (NOT (OTP X))
             (EQUAL X (ET))
             (AND (NOT (NUMBERP X2)) (EQUAL X (OT X1 X2 X3))))
         EQUAL (LABEL X) (ZERO)))
(IMPLIES (OR (NOT (OTP X))
             (EQUAL X (ET))
             (AND (NOT (OTP X3)) (EQUAL X (OT X1 X2 X3))))
         (EQUAL (RT X) (ET)))
(NOT (OTP T))
(NOT (OTP F))
(IMPLIES (OTP X) (NOT (r' X)))   ;; r' previously introduce recognizer
```

2b. There must be a well-founded relation r and a function m such that

```
(IMPLIES (AND (OTP X) (NOT (EQUAL X (ET))))  (r (m (LT X)) (m X)))
(IMPLIES (AND (OTP X) (NOT (EQUAL X (ET))))  (r (m (RT X)) (m X)))
```

are proveable. Take `m = COUNT`, `r = LESSP`.

2c. Let `(p L X) =`

```
(IMPLIES (AND (NUMBERP L) (ORDERED.TREE X))
              (ORDERED.TREE (INSERT L X)))
```

To instantiate the induction principal it is sufficient to find terms q, sX1, sL1, sX2, sL2 with at most L and X free, a well-founded relation r, and a measure m meeting the conditions of the induction principle that

```
(IMPLIES q (r (m sL1 sX1) (m L X)))
(IMPLIES q (r (m sL2 sX2) (m L X)))
(IMPLIES (NOT q) (p L X))
(IMPLIES (AND q (p sL1 sX1)  (p sL2 sX2)) (p L X))
```

are provable. This is satisfied by taking r and m as in B (m now ignoring its first argument) and

```
q =  (AND (OTP X) (NOT (EQUAL X (ET))))
sL1 = sL2 = L
sX1 = (LT X)
sX2 = (RT X)
```

Problem 3 (10 points). Distinct prefixes of s are inequivalent, so they must go to distinct states, requiring at least $L+1$ states. If the state encodes the longest prefix of s that is a suffix of the already-consumed input, $L+1$ suffices.

Problem 4 (10 points).

2a. [4 pts]

$$\{r \geq 0 \ \wedge \ \epsilon > 0\} \ P \ \{\sqrt{r} - \epsilon < z \leq \sqrt{r}\}.$$

2b. [6 pts] The inductive assertion [3 pts] is given by

$$(z \leq r) \ \wedge \ [(\sqrt{r} < z+v) \ \vee \ (r = z+v = 1)]$$

and the well-founded set and partial function [3 pts] are $(\mathcal{N}, <)$ and $\lfloor \frac{v}{\epsilon} \rfloor$, respectively.