Computer Science Department
Stanford University
Comprehensive Examination in Numerical Analysis
Autumn 1989

November 29, 1989

READ THIS FIRST!

1. You should write your answers for this part of the Comprehensive Examination in a BLUE BOOK. Be sure to write your MAGIC NUMBER on the cover of every blue book that you use.

2. The number of POINTS for each problem indicates how elaborate an answer is expected. For example, an essay-type question worth 6 points or less doesn't deserve an extremely detailed answer, even though a person can expound at length on just about any topic in computer science.

3. The total number of points is 30, and the exam takes 30 minutes. This "coincidence" can help you plan your time.

4. This exam is CLOSED BOOK. You may not use any notes, calculators, computers, or outside help.

5. Show your work, since PARTIAL CREDIT will be given for incomplete answers. For example, you can get credit for making a reasonable start on a problem even if the idea doesn't work out; you can also get credit for realizing that certain approaches are incorrect. On a true/false question, you might get partial credit for explaining why you think something is true when it is actually false. But no partial credit can be given if you write nothing.

Comprehensive Solutions: Numerical Analysis (30 points) Autumn 1989

Problem 1 (14 points).

1a. (4 points). Writing equations in matrix form:

$$Au = f.$$

Since

$$\det|A| = 1 - ab,$$

$\det|A| \neq 0$ when $ab \neq 1$. Thus the LU factorization exists.

1b. (5 points). We can write the iteration as:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = C \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{1}$$

where

$$C = \begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix}.$$

In addition, we have

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = C \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{2}$$

Subtracting (2) from (1), yields

$$e_{k+1} = Ce_k \tag{3}$$

where

$$e_k = \begin{pmatrix} x_k - \xi \\ y_k - \eta \end{pmatrix}.$$

Thus,

$$e_{k+2} = C^2 e_k.$$

Finally, taking norms gives

$$\| e_{k+1} \| \leq \| C^2 \| \; \| e_k \|.$$

Since $|ab| < 1$, $\| C^2 \| < 1$ and thus the iteration converges.

1c. (5 points). Once again we can write the iteration in matrix form as:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = C \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 1 \\ 2 - b \end{pmatrix}.$$

where this time

$$C = \begin{pmatrix} 0 & -a \\ 0 & ab \end{pmatrix}.$$

It is possible to show that for this matrix C, (3) holds and that the norm of $\| C \| < 1$.

Problem 2 (16 points).

2a. (4 points). Expand $f(x)$ about midpoint yields:

$$f(x) = f(y) + (x-y)f'(y) + \frac{1}{2}(x-y)^2 f''(y) + \cdots .$$

where

$$y = a + \frac{h}{2}.$$

Integrate series and simplify:

$$\int_a^{a+h} f(x)\,dx = hf(y) + \int_a^{a+h} (x-y)f'(y)\,dx + \frac{1}{2}\int_a^{a+h} (x-y)^2 f''(y)\,dx + \cdots .$$

Substitute $z = x - a - \frac{h}{2}$ on the righthand side:

$$\int_a^{a+h} f(x)\,dx = hf(y) + \int_{-\frac{h}{2}}^{\frac{h}{2}} zf'(y)\,dz + \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 f''(y)\,dz + \cdots .$$

Finally, we get:

$$\int_a^{a+h} f(x)\,dx = hf(y) + \frac{h^3}{24}f''(\eta)$$

where $\eta \in [a, a+h]$.

2b. (4 points). Summing the error terms:

$$\begin{aligned}
E &= \frac{1}{24}\sum_{i=1}^{n} h^3 f''(\eta_i) \qquad \eta_i \in [a+ih-h, a+ih+h] \\
&= \frac{1}{24}\sum_{i=1}^{n} h^3 f''(\eta) \qquad \eta \in [a, a+b] \\
&= \frac{1}{24} nh^3 f''(\eta) \\
&= \frac{(b-a)}{24} h^2 f''(\eta).
\end{aligned}$$

2c. (4 points). Must show that the error expression is positive

$$f''(x) = 6x^{-4} \quad > 0.$$

$$h^2 > 0.$$

$$b - a = 1 \quad > 0.$$

2d. (4 points). Determine n such that error expression above is less than 10^{-3}.

$$\max_{\eta\in[1,2]} f''(\eta) = 6.$$

So

$$n = \sqrt{\frac{6}{24 \times 10^{-3}}}.$$

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