Comprehensive: Analysis of Algorithms (80 points) **Autumn 1989**

The list of functions that are considered closed form includes $\lfloor x \rfloor$, $\lceil x \rceil$, F_n, H_n, $n!$, $n^{\underline{k}}$, $n^{\overline{k}}$, $\binom{n}{k}$.

Problem 1 (20 points). The generating function for the elements of Pascal's triangle along a line of slope $-1/2$ is given by

$$g_n(z) = \sum_k \binom{n-k}{n-2k} z^k ,$$

where $n \geq 0$.

1a. (7 points). Give a recurrence for $g_n(z)$ in terms of $g_{n-1}(z)$ and $g_{n-2}(z)$.

1b. (13 points). Evaluate $g_n(3)$ in closed form.

Problem 2 (20 points). Let $d(k)$ be the number of divisors of k (e.g., $d(10) = 4$).

2a. (7 points). Express

$$\sum_{k=1}^{n} d(k)$$

as $g(n) + O(n)$, with $g(n)$ in closed form.

2b. (13 points). Reduce the $O(n)$ error term from part (a) to $O(\sqrt{n})$. [*Hint:* Consider divisors smaller than $\sqrt{n}$ and greater than $\sqrt{n}$ separately.]

Problem 3 (11 points). Given n, exhibit an arithmetic progression $ai + b$ ($a \neq 0$) such that $\gcd(ai + b, aj + b) = 1$ for all $0 \leq i < j < n$.

Problem 4 (29 points). A binary search tree T is constructed with the elements of the set $[n] = \{1, 2, \ldots, n\}$. The elements of this set are inserted in T one by one, in the order given by a permutation of $[n]$ chosen uniformly at random from the set of all such permutations. [To insert a new element i in a tree: if the tree is empty, let i be the root. Otherwise compare i with the root j, and insert i in the left subtree or in the right subtree depending on whether $i < j$ or $i > j$.]

4a. (13 points). Show that the probability that element i is compared with element j when i is inserted in T is

$$p_{ij} = \frac{1}{|i-j| + 1} .$$

[*Hint:* How does i reach j?]

4b. (7 points). Determine the expected depth of element i in T.

4c. (9 points). What is the expected depth in T of an element chosen uniformly at random from $[n]$?

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