Comprehensive Solutions: Analysis of Algorithms (80 points) **Autumn 1989**

What you needed to know to get a passing score on the Analysis of Algorithms comprehensive:

For Problem 1a, in addition to the meaning of generating functions, and the additive recurrence for binomial coefficients, you needed to know that $\sum_{P(k)} f(k) = \sum_{P(k+c)} f(k+c)$. This is a routine transformation on sums, emphasized in *Concrete Mathematics*.

For Problem 2a, you needed to know another basic summation transform: Instead of

$$\sum_i [i \le n] \sum_d [d \text{ divides } i],$$

reverse the summation order as

$$\sum_d \sum_i [d \text{ divides } i \le n],$$

and the inner sum is then $\lfloor n/i \rfloor$.

For Problem 3, to get substantial partial credit and have a good chance of recognizing one of the many solutions, you needed to know that $p \perp q \equiv p \perp q + mp$. Then

$$aj + b \perp ak + b \equiv aj + b \perp a(k - j).$$

Some particular cases impose substantial constraints:

$$(j = 0) \quad b \perp ak \Rightarrow b \perp k$$

so b has no prime factors $\le n$. Then candidate values for b might be large primes, $\text{lcm}(2..n) + 1$, $n! + 1$, or simply 1.

For Problem 4b, you needed to know that the expected number of p_j that are true is the sum of the probabilities that each is true. This is a very important special case of the law that the expected value of a sum is the sum of the expected values. You also needed to recognize that a sum of reciprocals is a harmonic number.

For Problem 4c, you needed to know the meaning of "expected value", and how to sum harmonic numbers. Since harmonic numbers are themselves sums, expanding into a bivariate sum and reversing the summation order is routine.

Using the above knowledge, you could get

1a	7
1b	–
2a	7
2b	–
3	4
4a	–
4b	7
4c	9
	—
	34

which is a passing grade, by turning the crank.